

Heterogeneity-robust granular instruments*

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Abstract

Granular instrumental variables (GIV) has experienced sharp growth in empirical macro-finance. The methodology's rise showcases granularity's potential for identification across many economic environments, like the estimation of spillovers and demand systems. I propose a new estimator—called robust granular instrumental variables (RGIV)—that enables studying unit-level heterogeneity in spillovers. Unlike existing methods that assume heterogeneity is a function of observables, RGIV leaves heterogeneity unrestricted. In contrast to the baseline GIV estimator, RGIV allows for unknown shock variances and equal-sized units. Applied to the Euro area, I find strong evidence of country-level heterogeneity in sovereign yield spillovers.

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1 Introduction

In many macroeconomic settings, researchers wish to estimate the spillover of an idiosyncratic shock to one unit onto other units. Examples include estimating the contagion effects of financial market distress, aggregate demand externalities on individual consumption, strategic complementarities in price setting, and the price elasticity of demand of the stock market (Allen et al., 2009; Auclert et al., 2023; Alvarez et al., 2022; Gabaix and Koijen, 2021). Identifying plausibly exogenous variation in these settings however is notoriously difficult and contributes to the continued challenges of conducting empirical work.

Gabaix and Koijen (2020) tackles this difficulty with a novel technique, called granular instrumental variables (GIV). They consider environments where an individual unit’s outcome is partially determined by the size-weighted outcome. Thus, idiosyncratic shocks to a single unit spill over to all other units in equilibrium. To overcome the resulting endogeneity bias, their instrument for the size-weighted outcome is constructed as the difference between size- and equal-weighted outcomes. Their instrument is “granular” in that idiosyncratic shocks to large units are the primary source of identifying variation as they disproportionately contribute to movements in the size-weighted outcome variable. The broad adoption of GIV in applied macroeconomics and finance is indicative of its usefulness (Chodorow-Reich et al., 2021; Adrian et al., 2022; Camanho et al., 2022; Gabaix and Koijen, 2021).

The baseline GIV estimator of Gabaix and Koijen (2020), however, requires strong assumptions to satisfy the instrumental variables relevance condition and exclusion restriction and its extensions require further conditions. The baseline estimator requires homogeneous spillovers across units, homogeneous shock variances, skewed unit size, and idiosyncratic shocks (uncorrelated shocks between units after accounting for a *known* factor structure). In particular, there is typically no ex-ante rationale for there to be homogeneous spillovers and shock variances across units, contributing to the gap between theory and practice. Furthermore, I show that there is no general guarantee that the GIV spillover estimand admits a positive-weighted average of unit-specific spillovers. To address these concerns, Gabaix and Koijen (2020) generalize their procedure to allow for heterogeneity that is a function of observables, but the empirically important question of how to best tackle unobserved heterogeneity remains open.

This paper’s main contribution is to establish global identification for a GIV model with unit-specific spillovers and unknown shock variances. I propose an estimator, called robust granular instrumental variables (RGIV), that is robust in the sense that it is applicable to a wider set of environments than the baseline Gabaix and Koijen (2020) GIV estimator. Ultimately, RGIV brings the study of unit-level heterogeneity—a feature that is of substantial

importance in other areas of macroeconomics¹—to the estimation of spillovers.

RGIV uses internally estimated idiosyncratic shocks as instruments for the size-weighted outcome variable. Informally, the procedure can be described sequentially. An initial guess of the first unit’s spillover gives an estimate of the first unit’s idiosyncratic shock. Using this estimated shock as an instrument for the size-weighted outcome variable, the remaining units’ spillover coefficients and estimated idiosyncratic shocks can be computed. If one or more pairs of estimated idiosyncratic shocks are correlated, we can guess a new spillover for the first unit and repeat the procedure until the estimated idiosyncratic shocks are uncorrelated. RGIV makes use of granularity by exploiting the contribution of individual shocks (even to relatively small units) to movements in the size-weighted outcome variable. This approach differs from the baseline GIV estimator in that skewness in the size distribution of units and homogeneity in shock variances are not required.

I prove that there is an inevitable trade-off in assumptions between allowing for general spillover heterogeneity and a general shock covariance structure. Like GIV, the RGIV estimator can easily accommodate cross-unit correlations that are due to observed covariates (both time-varying and unit-specific) or a known factor structure in the residuals. However, I prove that a model with an *unknown* factor structure and unrestricted spillover heterogeneity is not identified. Hence, researchers must restrict either spillover heterogeneity or the correlation structure of the shocks.

I develop tests that evaluate the appropriateness of (1) the RGIV framework and (2) the homogeneous spillovers assumptions featured in Gabaix and Koijen (2020). These hypothesis tests are applications of standard GMM results (Newey and McFadden, 1994). The first test is the Sargan–Hansen test, where the null hypothesis of correct specification is rejected when one or more pairs of estimated idiosyncratic shocks are correlated. The second test is the distance metric test, where the null hypothesis is rejected when the constraint of spillover homogeneity binds.

Building on the analysis of a working paper version of Gabaix and Koijen (2020), I apply RGIV to sovereign yield spillovers in the Euro area and find strong evidence of spillover heterogeneity among Euro area members. I control for correlations induced by heterogeneous loadings on unobserved aggregate shocks by including a rich set of observed explanatory variables. In addition, I account for the well-documented correlation of shocks among Euro area countries by size-aggregating countries into larger “core” and “periphery” blocks (Bayoumi and Eichengreen, 1992). Under the preferred specification, I fail to reject the null hypothesis of correct specification. Meanwhile, the null hypothesis of spillover coefficient

¹For instance, take household-level differences in the marginal propensity to consume akin to Fagereng et al. (2021); Lewis et al. (2021); Fuster et al. (2021).

homogeneity is strongly rejected. I find that the size-weighted spillovers of countries in the western-periphery block (Portugal, Spain, and Ireland) are approximately twice that of the core country block.

In a simulation study, I also show that RGIV confidence intervals have good finite sample coverage properties under a DGP taken from the empirical application. GIV confidence intervals in contrast can undercover under spillover heterogeneity and when shock variances are taken to be unknown.

LITERATURE My model generalizes the baseline environment of Gabaix and Koijen (2020) to allow for spillover heterogeneity, shock variance heterogeneity, and equal-sized units. For estimation, RGIV fully exploits second moment information through GMM and can also be adapted to the case in which correlations in shocks are induced by factors with known loadings. In contrast, the spillover coefficient homogeneity, skewed size, and known shock variance conditions are used to justify the validity of the GIV instrument. Gabaix and Koijen (2020) also propose extensions that separately estimate spillover coefficient heterogeneity² and unknown shock variances³. In contrast, RGIV doesn't require skewness of the size distribution, heterogeneity to depend on observables, or for shock variances to be known; uncorrelated shocks is sufficient for jointly allowing unit-level spillovers and unknown shock variances.

Baumeister and Hamilton (2023) applies the insights of GIV to show that a rich, structural model of the oil market (with unobserved explanatory variables and inventories) can be estimated using full information maximum likelihood. Their estimator similarly exploits shock orthogonality (in their case orthogonality of supply and demand shocks) and also allows for elasticity heterogeneity. In contrast, I extend the simple setting originally considered by Gabaix and Koijen (2020) to transparently illustrate the conditions needed for identification, for non-identification with unobserved explanatory variables, and to characterize the RGIV estimator's limit behavior. Rather than focusing on modeling assumptions that are specific to particular applications, this paper seeks to provide researchers with a parsimonious framework that could be adapted to their specific applications.

Banafti and Lee (2022) propose a refinement to the GIV estimator, allowing for unobserved explanatory variables but require the number of economic units to be large and the

²Gabaix and Koijen (2020) Proposition 6 considers the case where unit-level spillover heterogeneity is determined by observables, Section D.9.1 proposes unit-specific "leave-one-out" granular instruments for when shock variances are known, and Section D.9.2 lists moment conditions for adapting GIV to heterogeneous spillovers though without the analysis of its econometric properties.

³Gabaix and Koijen (2020) Section D.3 appends shock variance moments after assuming homogeneous spillovers and global identification.

size distribution to be skewed, unlike RGIV. Here, the principal challenge is to prevent the law of large numbers from averaging away the granularity of units. The authors overcome this challenge by requiring the right tail of the unit size distribution to be highly skewed, with a Pareto tail index of less than 1. In contrast, RGIV is consistent even when the size distribution is uniform.

GIV procedures are closely linked to models found in the spatial panel econometric literature (Su et al., 2023; Aquaro et al., 2021; Chen et al., 2022). GIV’s spillover network structure can be viewed as a restricted form of spatial autocorrelation. Unlike these papers, I allow the number of units to be small (finite), show that the GIV spillovers under unit-level heterogeneity are globally identified, and estimate spillovers without the use of external instruments. More broadly, my procedure can also be viewed as a special case of a simultaneous equation model with covariance restrictions, which is closely related to impact matrix identification in the structural VAR setting (Sims, 1980; Hausman, 1983).

OUTLINE Section 2 illustrates the properties of GIV and RGIV in a simple three unit setting. Section 3 presents the main consistency and asymptotic normality results for the RGIV estimator. Section 4 describes the RGIV specification test and parameter homogeneity tests. Section 5 extends RGIV to include observable explanatory variables and discusses non-identification under unobserved explanatory variables. Section 6 applies the RGIV estimator to investigate sovereign yield spillovers in the Euro area. Section 7 examines finite-sample coverage accuracy of RGIV and GIV through simulations. Section 8 concludes.

2 Simple illustration

The remainder of this paper will consider the sovereign yield spillovers application as a running example. An idiosyncratic shock to one Euro area country raises yields in that country, as well as those of other Euro area countries since losses from default are partially shared. Country-specific spillovers are permitted.

In this section, I use a simple three-country setting to illustrate the construction and properties of GIV and RGIV. Section 3 formalizes the discussion and generalizes to n countries.

NOTATION Throughout this paper, I will follow the notation of Gabaix and Koijen (2020) for convenience. For a vector $X = (X_i)_{i=1,\dots,n}$ and size S_i satisfying $\sum_{i=1}^n S_i = 1$, the equal-

weighted sum X_E and size-weighted sum X_S are defined below:

$$X_E \equiv \frac{1}{n} \sum_{i=1}^n X_i, \quad X_S \equiv \sum_{i=1}^n S_i X_i.$$

Similarly, the equal-weighted and size-weighted cross-sectional averages for time series data $X_t = (X_{it})_{i=1, \dots, n}$ are defined as $X_{Et} \equiv \frac{1}{n} \sum_{i=1}^n X_{it}$ and $X_{St} \equiv \sum_{i=1}^n S_i X_{it}$ respectively.

2.1 Baseline three country setting

I will outline my “baseline” setting for the three country case using sovereign yield spillovers in the Euro area as a running example. For countries $i = 1, 2, 3$, let y_{it} be the yield spread relative to some comparison country. Yield spread growth is $r_{it} = \frac{y_{it} - y_{i,t-1}}{y_{i,t-1}}$. Size S_i is observed and corresponds to a country’s “debt at risk.” Here, $S_i \in (0, 1)$ is taken to be time-invariant and sums to 1.

Suppose the researcher is interested in estimating elasticity ϕ_i , which will be called the “spillover coefficient.” Specifically, yield spread growth r_{it} is determined by a country-specific spillover coefficient ϕ_i , size-aggregated yield spread r_{St} , and idiosyncratic shock u_{it} :

$$r_{it} = \phi_i r_{St} + u_{it}, \quad i = 1, \dots, n, \quad \phi_S < 1. \quad (1)$$

The size-weighted spillover coefficient ϕ_S is taken to be less than 1. Moreover, shocks u_{it} are mutually and serially independent, mean zero, and have country-specific variance $\mathbb{E}(u_{it}^2) = \sigma_i^2 > 0$.

The propagation of an idiosyncratic shock depends on the size of the shock’s origin country, the recipient country’s spillover coefficient, and the size-weighted average spillover coefficient. To illustrate, consider a unit idiosyncratic shock to country 1 ($u_{1t} = 1$) holding the idiosyncratic shocks of countries 2 and 3 at zero ($u_{2t} = u_{3t} = 0$). Then, the size-weighted yield spread growth r_{St} increases by $S_1 \times \frac{1}{1 - \phi_S}$, which is computed from taking a size-weighted average of Equation 1:

$$\sum_{i=1}^3 S_i r_{it} = \sum_{i=1}^3 S_i (\phi_i r_{St} + u_{it}) \implies r_{St} = \frac{u_{St}}{1 - \phi_S}. \quad (2)$$

The increase in r_{St} is larger when the size of country 1 is large and when countries are, on average, sensitive to spillovers (from multiplier $\frac{1}{1 - \phi_S}$). Then, idiosyncratic shock u_{1t} spills over to r_{2t} and r_{3t} , giving rise to increases of $\phi_j r_{St} = \phi_j \frac{S_1}{1 - \phi_S}$ for $j = 2, 3$. The total increase in r_{1t} is $1 + \phi_1 \frac{S_1}{1 - \phi_S}$, a composition of the direct effect of idiosyncratic shock u_{1t} and a spillover effect.

Granularity rules out the ordinary least squares regression of r_{it} on r_{St} as a method for estimating ϕ_i . Equation 2 highlights granularity in the baseline setting, showing that idiosyncratic shocks are responsible for movements in the aggregated yield spread r_{St} since $S_i > 0$ and $\sigma_i^2 > 0$. Therefore, regressing r_{it} on r_{St} to estimate ϕ_i as $T \rightarrow \infty$ suffers from endogeneity bias since $\mathbb{E}(r_{St}u_{1t}) = \frac{S_1\sigma_1^2}{1-\phi_S} \neq 0$. An alternative estimator is needed.

2.2 Illustration of GIV

In this section, I review the baseline GIV estimator of Gabaix and Koijen (2020) applied to this simple setting and discuss the conditions for its validity. I show that a skewed size distribution is necessary for the relevance condition to hold. Shock orthogonality and homogeneous idiosyncratic shock variances are necessary for the instrumental variables exclusion restriction to hold. Moreover, the exclusion restriction fails under heterogeneity of spillover coefficients across countries. As discussed later, there are GIV extensions that individually accommodate spillover coefficient heterogeneity and heterogeneous (and unknown) shock variances, but these require additional conditions.

Take the setting of Section 2.1 and further assume homogeneous spillover coefficients ($\phi_1 = \phi_2 = \phi_3 = \phi$), homogeneous shock variances ($\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$), and skewed unit sizes (ruling out $S_1 = S_2 = S_3$). Just as in Section 2.1, granularity induces endogeneity bias in the regression of r_{it} on r_{St} for regression coefficient $\hat{\phi}_{OLS}^i$

$$\hat{\phi}_{OLS}^i - \phi \xrightarrow{p} \frac{\mathbb{E}(r_{St}r_{it})}{\mathbb{E}(r_{St}^2)} - \phi = \frac{S_i\sigma^2}{\mathbb{E}(r_{St}^2)[1-\phi]} > \phi.$$

Gabaix and Koijen (2020) propose estimating ϕ using instrumental variables. An equal-weighted aggregation of the country-level yield spread growth gives rise to the following IV regression model

$$r_{Et} = \phi r_{St} + u_{Et}.$$

The equal-weighted yield spread growth responds to the size-weighted yield spread growth r_{St} (through spillover coefficient ϕ) and equal-weighted idiosyncratic shocks u_{Et} . For the IV regression model in the above display, “granular instrument” z_t is constructed as the difference between the size- and equal-weighted yield spread growth

$$z_t := r_{St} - r_{Et} = (\phi r_{St} + u_{St}) - (\phi r_{St} + u_{Et}) = u_{St} - u_{Et}.$$

From the homogeneous spillovers assumption, the endogeneity from the size-weighted yield

spread growth r_{St} is differenced away.

Skewness in country sizes is crucial for the instrumental variables relevance condition to hold and z_t reflects variation coming from idiosyncratic shocks to relatively large countries. To illustrate, first consider the extreme case of no skewness ($S_1 = S_2 = S_3$). Then the size- and equal-weighted yield spread growths are identical, producing a granular instrument that is identically zero $z_t = 0$. More generally, the instrumental variables relevance condition can be explicitly computed:

$$\begin{aligned}\mathbb{E}[z_t r_{St}] &= \frac{1}{1-\phi} \mathbb{E}[u_{St}(u_{St} - u_{Et})] \\ &= \frac{\sigma^2}{1-\phi} \left[S_1 \left(S_1 - \frac{1}{3} \right) + S_2 \left(S_2 - \frac{1}{3} \right) + S_3 \left(S_3 - \frac{1}{3} \right) \right].\end{aligned}$$

When the size of country 1 is large relative to other countries (when $S_1 \gg S_2, S_3$), the relevance condition is further from zero reflected by the $S_1(S_1 - 1/3)$ term.

The shock orthogonality and homogeneous (or known) shock variance assumptions are crucial for the exclusion restriction to hold. Explicitly, the covariance between the instrument and IV regression model error u_{Et} is

$$\begin{aligned}\mathbb{E}[u_{Et}(u_{St} - u_{Et})] &= \frac{1}{3} [S_1\sigma^2 + S_2\sigma^2 + S_3\sigma^2] - \frac{1}{3} \left[\frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2 + \frac{1}{3}\sigma^2 \right] \\ &= \frac{1}{3}\sigma^2 - \frac{1}{3}\sigma^2 = 0.\end{aligned}$$

In the first line of the above display, the uncorrelated shock assumption ensures that covariance terms $\mathbb{E}(u_{it}u_{jt})$ are zero. The homogeneous shock variance assumption ensures that the difference between the first line's bracketed terms is zero. Note that any instrument of the form $z'_t = W_1 r_{1t} + W_2 r_{2t} + W_3 r_{3t} - \frac{1}{3} r_{1t} - \frac{1}{3} r_{2t} - \frac{1}{3} r_{3t}$ where $W_1 + W_2 + W_3 = 1$ satisfies the instrumental variables exclusion restriction. Gabaix and Koijen (2020) show that weighting by size gives a variance-minimizing estimator for the IV regression model. The spillover coefficient can also be consistently estimated when a time fixed effect is included, as the time fixed effect is differenced away in the construction of z_t .

The exclusion restriction argument described in the preceding paragraph can easily be extended to settings in which the shock variances are known to the econometrician rather than being homogeneous across units. Gabaix and Koijen (2020) show that identical computations hold after replacing the equal weights in $z_t = r_{St} - r_{Et}$ with inverse variance weights.

Spillover coefficient heterogeneity leads to a failure in the instrumental variables exclusion restriction. Allowing for unit-specific spillover coefficients, the granular instrument contains

variation from r_{St} :

$$z_t = r_{St} - r_{Et} = (\phi_S - \phi_E)r_{St} + u_{St} - u_{Et}. \quad (3)$$

From the $(\phi_S - \phi_E)$ term, the magnitude of the exclusion restriction's violation is greater when spillover coefficients vary systematically by size. Moreover, I show later in Proposition 1 that there is no guarantee that the GIV spillover coefficient estimand is a non-negative weighted average of spillover coefficients. In this sense, the GIV spillover coefficient estimand could be far from the potentially heterogeneous true spillover coefficients.

2.3 RGIV illustration: Environment as GMM moment conditions

This subsection introduces the estimator proposed in this paper, robust granular instrumental variables (RGIV). RGIV exploits the uncorrelatedness of idiosyncratic shocks through the generalized method of moments. The estimator's identifying variation comes from individual country-level idiosyncratic shocks.

Returning to the baseline environment described in Section 2.1—which again features unit-specific spillover coefficients ϕ_i and unit-specific shock variances σ_i^2 —the RGIV estimator encodes the uncorrelatedness of idiosyncratic shocks through GMM moment conditions. Taking sizes S_1, S_2, S_3 as given, store data in the vector $\mathbf{r}_t = [r_{1t}, r_{2t}, r_{3t}]'$ and parameters in $\boldsymbol{\phi} = [\phi_1, \phi_2, \phi_3]'$. Letting $u_i(\mathbf{r}_t, \boldsymbol{\phi}) = r_{it} - \phi_i r_{St}$, moment function $g(\mathbf{r}_t, \boldsymbol{\phi})$ encodes the condition that idiosyncratic shocks are uncorrelated

$$g(\mathbf{r}_t, \boldsymbol{\phi}) = \left[u_1(\mathbf{r}_t, \boldsymbol{\phi})u_2(\mathbf{r}_t, \boldsymbol{\phi}) \quad u_1(\mathbf{r}_t, \boldsymbol{\phi})u_3(\mathbf{r}_t, \boldsymbol{\phi}) \quad u_2(\mathbf{r}_t, \boldsymbol{\phi})u_3(\mathbf{r}_t, \boldsymbol{\phi}) \right]'$$

where $\mathbb{E}[g(\mathbf{r}_t, \boldsymbol{\phi}_0)] = 0$ for true parameter $\boldsymbol{\phi}_0$. The parameter-dependent weight matrix is $\widehat{W}(\boldsymbol{\phi}) = \text{diag}\left(\frac{1}{\widehat{\sigma}_1^2(\boldsymbol{\phi})\widehat{\sigma}_2^2(\boldsymbol{\phi})}, \frac{1}{\widehat{\sigma}_1^2(\boldsymbol{\phi})\widehat{\sigma}_3^2(\boldsymbol{\phi})}, \frac{1}{\widehat{\sigma}_2^2(\boldsymbol{\phi})\widehat{\sigma}_3^2(\boldsymbol{\phi})}\right)$ for $\widehat{\sigma}_i^2(\boldsymbol{\phi}) = \frac{1}{T} \sum_{t=1}^T u_i(\mathbf{r}_t, \boldsymbol{\phi})^2$. Then, the robust granular instrumental variables (RGIV) estimator is defined as a continuously updating GMM estimator:

$$\widehat{\boldsymbol{\phi}}^{RGIV} = \arg \min_{\boldsymbol{\phi}: \phi_S < 1} \left(\frac{1}{T} \sum_{t=1}^T g(\mathbf{r}_t, \boldsymbol{\phi}) \right)' \widehat{W}(\boldsymbol{\phi}) \left(\frac{1}{T} \sum_{t=1}^T g(\mathbf{r}_t, \boldsymbol{\phi}) \right). \quad (4)$$

$\widehat{W}(\boldsymbol{\phi})$ is an efficient GMM weight matrix when idiosyncratic shocks are independent. Avoiding inversion of a potentially non-diagonal matrix, $\widehat{W}(\boldsymbol{\phi})$ also ensures numerical stability if the initialization of $\boldsymbol{\phi}$ is far from the GMM objective function's minimum.

Equivalently, the optimization problem characterized in Equation 4 minimizes the average squared correlation coefficients between pairs of estimated idiosyncratic shocks. To

see this, define the estimated correlation coefficient of idiosyncratic shocks as $\hat{\rho}_{ij}(\boldsymbol{\phi}) = \frac{\frac{1}{T} \sum_{t=1}^T u_i(\mathbf{r}_t, \boldsymbol{\phi}) u_j(\mathbf{r}_t, \boldsymbol{\phi})}{\sqrt{\hat{\sigma}_i^2(\boldsymbol{\phi}) \hat{\sigma}_j^2(\boldsymbol{\phi})}}$. Then, the RGIV estimator is

$$\hat{\boldsymbol{\phi}}^{RGIV} = \arg \min_{\boldsymbol{\phi}: \phi_S < 1} \frac{1}{3} \left[\hat{\rho}_{12}(\boldsymbol{\phi})^2 + \hat{\rho}_{13}(\boldsymbol{\phi})^2 + \hat{\rho}_{23}(\boldsymbol{\phi})^2 \right]$$

after multiplying the objective function in Equation 4 by $\frac{1}{3}$. Intuitively, RGIV chooses the spillover coefficient vector $\boldsymbol{\phi}$ that makes the estimated shocks the least correlated.

RGIV also admits an instrumental variables interpretation, as a country's spillover coefficient is estimated using information from internally estimated idiosyncratic shocks to other countries. The asymptotic variance matrix of the RGIV estimator is $V = (G' \Sigma^{-1} G)^{-1}$ for Jacobian matrix $G = \mathbb{E}[\nabla_{\boldsymbol{\phi}} g(\mathbf{r}_t, \boldsymbol{\phi}_0)]$ and moment covariance matrix $\Sigma = \mathbb{E}[g(\mathbf{r}_t, \boldsymbol{\phi}_0) g(\mathbf{r}_t, \boldsymbol{\phi}_0)']$. Then the diagonal entries of V are

$$\text{Avar} \left(\hat{\phi}_i^{RGIV} \right) = \frac{\sigma_i^2}{\prod_{j \neq i} (S_j^2 \sigma_j^2)} \frac{(1 - \phi_S)^2 (S_1^2 \sigma_1^2 + S_2^2 \sigma_2^2 + S_3^2 \sigma_3^2)}{4}.$$

The above display illustrates that the identifying variation of the RGIV estimator doesn't require skewness in the unit size distribution, as the identifying variation comes from individual idiosyncratic shocks. Estimates for $\hat{\phi}_i^{RGIV}$ are more precise when idiosyncratic shocks to countries $j \neq i$ have higher variance. Intuitively, the estimated idiosyncratic shocks can be viewed as sequentially estimated internal instruments. Guessing the spillover coefficient of country 1, the resulting estimated idiosyncratic shock to country 1 can be used as an instrument for the estimation of the spillover coefficients for countries 2 and 3. If one or more pairs of estimated idiosyncratic shocks are too correlated, the procedure is repeated for a new spillover coefficient.

3 General model and main results

This section describes the assumptions needed for the robust GIV estimator for n countries.

3.1 Assumptions

Assumption 1. (*Baseline model*)

- (i) **Model:** For $n \geq 3$ units, let known sizes $S_i \in (0, 1)$ sum to 1. Outcome $\mathbf{r}_t = [r_{1t}, \dots, r_{nt}]'$ responds to the size-aggregated outcome r_{St} according to spillover coef-

ficient ϕ_i and unobserved shocks $\mathbf{u}_t = [u_{1t}, u_{2t}, \dots, u_{nt}]'$

$$r_{it} = \phi_i r_{St} + u_{it}, \quad \forall i = 1, \dots, n, \quad \phi_S < 1.$$

(ii) **Shock moments:** For $\sigma_i^2 > 0$, shocks \mathbf{u}_t are i.i.d. with moments $\mathbb{E}(\mathbf{u}_t) = 0$, $\mathbb{E}(\mathbf{u}_t \mathbf{u}_t') = \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$, and $\mathbb{E}(\|\mathbf{u}_t\|^4) < \infty$. Moreover, u_{it} is independent of u_{jt} for $i \neq j$.

(iii) **Parameter space:** For spillover coefficient $\boldsymbol{\phi} = [\phi_1, \dots, \phi_n]'$, the true parameter $\boldsymbol{\phi}_0$ is in the interior of parameter space Φ . Φ is compact and for any $\tilde{\boldsymbol{\phi}} \in \Phi$, $\tilde{\phi}_S < 1$.

In Assumption 1(i), the outcome variable r_{it} responds to the size-aggregated outcome r_{St} according to spillover coefficient ϕ_i and idiosyncratic shock u_{it} . Through $\phi_S < 1$, positive idiosyncratic shocks increase the size-weighted outcome r_{St} . As will be outlined below, the RGIV estimator is just-identified for $n = 3$ and over-identified for $n > 3$. Moreover, size S_i is taken to be *known* by the econometrician. Hence, the model presented in 1(i) can be modified to allow for time-varying size (for $S_{it} \in (0, 1)$ and $\sum_{i=1}^n S_{it} = 1$) without changing the proofs to follow. In Assumption 1(iii), the parameter space is assumed to be compact and restricted to encode the sign restriction of $\phi_S < 1$. Compactness is a standard technical assumption for the consistency of extremum estimators (Newey and McFadden, 1994).

3.2 RGIV estimator

The GMM estimator established in Definition 1 below (called RGIV) exploits the uncorrelated unit-specific shock condition established in Assumption 1(ii). RGIV is constructed as a continuously updating GMM estimator (Hansen et al., 1996). The moment function $g(\mathbf{r}_t, \boldsymbol{\phi})$ contains the pairwise products of each of the estimated shocks, and the weight matrix inversely weights each moment by the product of the respective estimated shock variances. Exploiting shock uncorrelatedness can be best understood as being consistent with the tradition of exploiting second moment information in the traditional SVAR identification setting (Kilian and Lütkepohl, 2017; Leeper et al., 1996).

Definition 1. For outcome variable $\mathbf{r}_t = [r_{1t}, \dots, r_{nt}]'$, let $u_i(\mathbf{r}_t, \boldsymbol{\phi}) = r_{it} - \phi_i r_{St}$ for $i = 1, \dots, n$. The moment function $g(\mathbf{r}_t, \boldsymbol{\phi})$ is

$$g(\mathbf{r}_t, \boldsymbol{\phi}) = [u_1(\mathbf{r}_t, \boldsymbol{\phi})u_2(\mathbf{r}_t, \boldsymbol{\phi}), \dots, u_1(\mathbf{r}_t, \boldsymbol{\phi})u_n(\mathbf{r}_t, \boldsymbol{\phi}), u_2(\mathbf{r}_t, \boldsymbol{\phi})u_3(\mathbf{r}_t, \boldsymbol{\phi}), \dots, u_{n-1}(\mathbf{r}_t, \boldsymbol{\phi})u_n(\mathbf{r}_t, \boldsymbol{\phi})]'$$

For $\hat{\sigma}_i^2(\boldsymbol{\phi}) = \frac{1}{T} \sum_{t=1}^T u_i(\mathbf{r}_t, \boldsymbol{\phi})^2$, define the sample weight matrix

$$\widehat{W}(\boldsymbol{\phi}) = \text{diag}\left(\frac{1}{\hat{\sigma}_1^2(\boldsymbol{\phi})\hat{\sigma}_2^2(\boldsymbol{\phi})}, \dots, \frac{1}{\hat{\sigma}_1^2(\boldsymbol{\phi})\hat{\sigma}_n^2(\boldsymbol{\phi})}, \frac{1}{\hat{\sigma}_2^2(\boldsymbol{\phi})\hat{\sigma}_3^2(\boldsymbol{\phi})}, \dots, \frac{1}{\hat{\sigma}_{n-1}^2(\boldsymbol{\phi})\hat{\sigma}_n^2(\boldsymbol{\phi})}\right).$$

Then, for GMM objective function $\widehat{Q}_T(\phi) = \left[\frac{1}{T} \sum_{t=1}^T g(\mathbf{r}_t, \phi) \right]' \widehat{W}(\phi) \left[\frac{1}{T} \sum_{t=1}^T g(\mathbf{r}_t, \phi) \right]$, the **robust granular instrumental variables (RGIV)** estimator is $\widehat{\phi}^{RGIV} = \arg \min_{\phi: \phi_S < 1} \widehat{Q}_T(\phi)$.

The RGIV estimator is robust in that the estimator allows for unit-specific spillover coefficient heterogeneity while also allowing for unknown shock variances and equal unit sizes. When the researcher is *a priori* certain of homogeneous spillover coefficients $\phi_i = \phi$, the RGIV moment vector can accommodate this parameter restriction by restricting the elements of parameter vector ϕ to be homogeneous across units yielding possible gains in efficiency. Moreover, the assumption of parameter homogeneity is formally testable as will be discussed in Section 4.

Lemma 1 (Identification). *Impose Assumption 1. For $g_0(\phi) = \mathbb{E}[g(\mathbf{r}, \phi)]$, $g_0(\phi_0) = 0$ for the true parameter ϕ_0 and $g_0(\check{\phi}) \neq 0$ for $\check{\phi} \in \Phi$ such that $\check{\phi} \neq \phi_0$.*

Proof. See Appendix A.4. □

For Lemma 1, the parameter space restriction $\check{\phi}_S < 1$ in Assumption 1(iii) rules out the false solution to the population moment condition. The proof of Lemma 1 shows that there is a second parameter $\check{\phi}$ such that $g_0(\check{\phi}) = 0$. However, $\check{\phi}$ is not in the parameter space because $\check{\phi}_S = 1 + (1 - \phi_S) > 1$ and thus is not a candidate solution. In economic terms, Assumption 1(iii) represents the researcher's knowledge of the sign of an idiosyncratic shock's effect on the size-weighted outcome r_{St} . This knowledge can come from an application's institutional details; for the case of the running example of Eurozone yield spread spillovers, a positive idiosyncratic shock to one country gives rise to an increase in aggregated yield spreads since losses from the default of government debt are partially shared. The identification lemma can also be adapted for the opposite case $\phi_S > 1$ by reversing the inequality specified in Assumption 1(iii) to $\check{\phi}_S > 1$.

Mimicking Assumption 1 of Gabaix and Koijen (2020), the Lemma is derived under the weaker condition that the correlation structure of shocks is known (see Condition (ii') in Appendix A.4 for details). Theorems 1 and 2 (below), which establish consistency and asymptotic normality of the RGIV estimator, are shown under the assumption of uncorrelated shocks but can analogously be adapted to the known shock correlation case. These results follow from a standard application of arguments provided in Pakes and Pollard (1989).

Theorem 1 (Consistency of RGIV). *Impose Assumption 1. The RGIV estimator is consistent $\widehat{\phi}^{RGIV} \xrightarrow{p} \phi_0$ for the true parameter ϕ_0 as $T \rightarrow \infty$.*

Proof. See Appendix A.1. □

Theorem 2 (Asymptotic normality of RGIV). *Impose Assumption 1. The RGIV estimator is asymptotically normal*

$$\sqrt{T}(\hat{\phi}^{RGIV} - \phi_0) \xrightarrow{d} \mathcal{N}(0, (G'WG)^{-1}G'W\Sigma WG(G'WG)^{-1})$$

for RGIV population weight matrix $W = \text{diag}(\frac{1}{\sigma_1^2\sigma_2^2}, \dots, \frac{1}{\sigma_1^2\sigma_n^2}, \frac{1}{\sigma_2^2\sigma_3^2}, \dots, \frac{1}{\sigma_{n-1}^2\sigma_n^2})$, moment covariance matrix $\Sigma = \mathbb{E}[g(\mathbf{r}_t, \phi_0)g(\mathbf{r}_t, \phi_0)']$ and $G = \mathbb{E}[\nabla_{\phi}g(\mathbf{r}_t, \phi_0)]$ as $T \rightarrow \infty$.

Proof. See Appendix A.2. □

REMARKS.

1. The assumptions of granularity ($S_i > 0$) and non-zero shock variances ($\sigma_i^2 > 0$) guarantee that $G'WG$ is full rank. Intuitively, these conditions guarantee that idiosyncratic shocks contribute to fluctuations in the endogenous variable r_{st} and can be exploited as identifying variation. Granularity and non-zero shock variances can loosely be viewed as the corresponding “relevance conditions” for the RGIV setting.
2. Diagonal weight matrix $\widehat{W}(\hat{\phi}^{RGIV})$ is an estimator for the efficient GMM weight matrix when idiosyncratic shocks are not only uncorrelated, but also independent. Its specific form allows the RGIV estimator to be interpreted as the ϕ that minimizes the average squared correlation coefficient between shocks. Practically, a continuously updating GMM estimator with diagonal weight matrix $\widehat{W}(\phi)$ is one way to ensure the stability of the GMM objective function when T is relatively small. The choice of a diagonal $\widehat{W}(\phi)$ hedges against potential numerical instability relative to choosing an alternative weight matrix constructed as $\overline{W}(\phi) = [\frac{1}{T} \sum_{t=1}^T g(\mathbf{r}_t, \phi)g(\mathbf{r}_t, \phi)']^{-1}$. Evaluating an objective function using a weight matrix of the form $\overline{W}(\phi)$ would require inverting the potentially ill-conditioned matrix $\frac{1}{T} \sum_{t=1}^T g(\mathbf{r}_t, \phi)g(\mathbf{r}_t, \phi)'$, which could arise if the initial conditions for the sample GMM objective function are far from the true value.
3. In Section 7, the procedure is implemented in MATLAB using the constrained non-linear optimizer `fmincon()`. The sample GMM objective function $\widehat{Q}_T(\phi)$ described in Definition 1 is used as the objective function with the constraint $\phi_S < 1$. As is standard for GMM estimators, the asymptotic variance for $\hat{\phi}^{RGIV}$ can be computed using “plug-in” estimators for the components of the asymptotic variance expression in Theorem 2. Specifically, I use the weight matrix estimator $\widehat{W}(\hat{\phi}^{RGIV})$ from Definition 1 for W , the sample analogue of the moment covariance matrix $\widehat{\Sigma} = \frac{1}{T} \sum_{t=1}^T g(\mathbf{r}_t, \hat{\phi}^{RGIV})g(\mathbf{r}_t, \hat{\phi}^{RGIV})'$ for Σ , and the sample analogue of the Jacobian matrix $\widehat{G} = \frac{1}{T} \sum_{t=1}^T \nabla_{\phi}g(\mathbf{r}_t, \hat{\phi}^{RGIV})$ for G .

4. Under the weaker condition of cross-sectional shock uncorrelatedness (imposing $\mathbb{E}[u_{it}u_{jt}] = 0$ for $i \neq j$ rather than the stronger condition of $u_{it} \perp\!\!\!\perp u_{jt}$), the RGIV estimator is still consistent and asymptotically normal. However, the GMM weight matrix $\widehat{W}(\widehat{\phi}^{RGIV})$ would no longer converge in probability to the efficient GMM weight matrix $W = \mathbb{E}[g(\mathbf{r}_t, \phi_0)g(\mathbf{r}_t, \phi_0)']^{-1}$. Shock independence guarantees that off-diagonal elements of W are zero and that the on-diagonal elements are multiplicatively separable i.e. that $\mathbb{E}(u_{it}^2 u_{jt}^2) = \mathbb{E}(u_{it}^2)\mathbb{E}(u_{jt}^2)$ for $i \neq j$. Under the weaker condition of uncorrelated shocks, higher order dependence (like that arising from a shared volatility term) is permissible.⁴ Here, the weight matrix is no longer efficient but the estimator is still consistent and asymptotically normal.
5. $\widehat{\phi}^{RGIV}$ can still be used as a plug-in estimator for the efficient GMM weight matrix when idiosyncratic shocks are uncorrelated but not independent. The following gives an example of one such procedure. First, as noted in the previous bullet, RGIV can be used to obtain a preliminary (consistent) estimate of the spillover coefficient. Second, GMM can be used to estimate $\widehat{\phi}^{\text{Step 2}}$ with sample GMM weight matrix $\widehat{W}^{\text{Step 2}} = [\frac{1}{T} \sum_{t=1}^T g(\mathbf{r}_t, \widehat{\phi}^{RGIV})g(\mathbf{r}_t, \widehat{\phi}^{RGIV})']^{-1}$. Since $\widehat{W}^{\text{Step 2}}$ is an estimator for the efficient GMM weight matrix under uncorrelated but not independent shocks, $\widehat{\phi}^{\text{Step 2}}$ is an efficient estimator for ϕ_0 .

3.3 GIV under spillover coefficient heterogeneity

In general, the baseline Gabaix and Koijen (2020) GIV estimand ϕ^{GIV} under spillover coefficient heterogeneity cannot be interpreted as a weighted average of unit-specific spillover coefficients. From Equation 3, the spillover coefficient homogeneity assumption is necessary for the instrumental variable exclusion restriction to hold, as it ensures the endogenous term r_{st} is differenced away. When shock variances are homogeneous across countries, Proposition 1 (below) decomposes the GIV estimand into an equal-weighted spillover coefficient term and a term that depends on $(\phi_S - \phi_E)$. The bias is larger when the spillover coefficient varies systematically with the size distribution, giving rise to a larger gap between ϕ_S and ϕ_E . Moreover, as also discussed in Appendix D.8 of Gabaix and Koijen (2020), the bias is smaller when the number of units is large.

⁴Serial correlation is another form of dependence of interest for applied work. Here, Theorem 2 can be adapted to serially correlated shocks by exchanging the currently used central limit theorem for independent and identically distributed data for one applicable to data with dependence, like those found in Davidson (1994).

Proposition 1. *Impose Assumption 1 and shock variance homogeneity ($\sigma_i^2 = \sigma^2$). Then the GIV estimand is*

$$\phi^{GIV} = \frac{\mathbb{E}(z_t r_{Et})}{\mathbb{E}(z_t r_{St})} = \phi_E + \frac{\phi_S - \phi_E}{n} \cdot \frac{1}{\frac{\phi_S - \phi_E}{1 - \phi_S} [\sum_{i=1}^n S_i^2] - \frac{1}{n} + \sum_{i=1}^n S_i^2}.$$

Proof. See A.3. □

Proposition 1 implies that the GIV estimand doesn't admit a weighted average interpretation of unit-specific spillover coefficients. To see this, consider the following example. Suppose $n = 3$, $\mathbf{S} = [0.2 \ 0.3 \ 0.5]'$, and $\phi = [0.6 \ 0.3 \ 0.3]'$. Applying Proposition 1, $\phi^{GIV} = -0.18 \notin [0.3, 0.6]$, so ϕ^{GIV} is not a positive weighted average of individual spillover coefficients. Section 7 investigates the practical implications of GIV under spillover coefficient heterogeneity using an empirically relevant DGP.

While Proposition 1 highlights the potential pitfalls from mistakenly applying the baseline GIV estimator, Gabaix and Koijen (2020) also give guidance for estimating unit-specific spillover coefficients under additional restrictions. Their Proposition 6 and Appendix D.9 present alternative procedures that require heterogeneity to depend on observables and for the shock variance to be known respectively. In contrast, these restrictions are unnecessary for RGIV.

4 Testing

In this section, I propose two tests derived from standard GMM results (Newey and McFadden, 1994): a test of over-identifying restrictions that evaluates the uncorrelatedness of idiosyncratic shocks and a test that evaluates the homogeneous spillover coefficient condition of Gabaix and Koijen (2020). For what follows, impose Assumption 1.

4.1 RGIV specification test

The uncorrelatedness of idiosyncratic shocks is directly testable when there are four or more units, as the number of moment conditions exceeds the number of unit-specific spillover coefficients. Recall that the RGIV estimator is a GMM estimator for moment function $g(\mathbf{r}_t, \phi)$, which encodes the pairwise uncorrelatedness of idiosyncratic shocks. For $n \geq 4$ countries, the number of moments exceeds the number of estimated parameters allowing for the use of the Sargan–Hansen test. The null hypothesis $H_0: \mathbb{E}[g(\mathbf{r}_t, \phi_0)] = \mathbf{0}$ for true parameter ϕ_0 is rejected for large values of the J -statistic $J_T = T \cdot \widehat{Q}_T(\widehat{\phi}^{RGIV})$. Intuitively, J_T is large when one or more pairs of estimated idiosyncratic shocks are correlated.

4.2 Spillover coefficient homogeneity test

Spillover coefficient homogeneity across units is not only a subject of potential substantive interest—particularly given its role in the Gabaix and Koijen (2020) GIV estimator—but also can be formally evaluated within the framework of RGIV. Since the null hypothesis of coefficient homogeneity $H_0: \phi_1 = \phi_2 = \dots = \phi_n$ is a special case of RGIV’s unit-specific spillover coefficients, H_0 can be tested with the distance metric test. Here, estimator $\bar{\phi}$ minimizes $\hat{Q}_T(\phi)$ subject to the constraints of null hypothesis H_0 . Then, spillover coefficient homogeneity is rejected when the distance metric test statistic $DM_T = T(\hat{Q}_T(\bar{\phi}) - \hat{Q}_T(\hat{\phi}^{RGIV}))$ is large since $DM_T \xrightarrow{d} \chi_{n-1}^2$ under the null hypothesis. A large value of DM_T indicates that the constraints of null hypothesis H_0 bind.

5 Extensions to additional explanatory variables

Motivated by the requirements of empirical applications, this section discusses two extensions that relax the condition of uncorrelated idiosyncratic shocks discussed in Section 3 through observed and unobserved explanatory variables. I show that when observable time-varying explanatory variables determine the shocks’ correlation structure, RGIV can be used after residualizing the outcome variables with respect to these observables. When the correlation structure is instead determined by unobserved factors with unknown loadings, the global identification condition fails.

5.1 RGIV with observed explanatory variables

In practice, observable characteristics can determine the correlation structure among shocks as described by the following two situations. First, the outcome variable could have unit-specific exposures to a particular observed variable—take country-specific exposures to the USD-EUR exchange rate in the Euro area sovereign yields example. Second, observable characteristics could also be used to account for a correlation structure driven by unobserved explanatory variables—like country-specific exposures to a “global financial conditions” factor—so long as such unobserved factors are in the span of the observed explanatory variables.

Observed variable \mathbf{x}_t ($k \times 1$) affects outcome variable r_{it} through a direct effect and an indirect effect. Modifying Assumption 1(i), unit-specific coefficients β_i determine the

cross-sectional correlation of v_{it}

$$r_{it} = \phi_i r_{St} + \underbrace{\beta_i' \mathbf{x}_t}_{v_{it}} + u_{it}, \quad \mathbf{x}_t \perp\!\!\!\perp u_{it}, \quad i = 1, \dots, n \quad (5)$$

where u_{it} is still idiosyncratic in the sense that u_{it} is independent of u_{jt} for $i \neq j$. The orthogonality condition $u_{it} \perp\!\!\!\perp \mathbf{x}_t$ can be interpreted as a “selection-on-observables” assumption; the cross-sectional correlation of v_{it} is entirely determined by the observed explanatory variables. Holding r_{St} and u_{it} constant, β_i can be interpreted as the direct effect of \mathbf{x}_t on r_{it} . Additionally, mediated through changes in $r_{St} = \frac{1}{1-\phi_S}[\beta_S' \mathbf{x}_t + u_{St}]$, $\frac{\phi_i}{1-\phi_S} \beta_S$ determines the indirect effect of \mathbf{x}_t on r_{it} . Analogous to the discussion in Section 2.1, the indirect effect of a change of \mathbf{x}_t on unit i is larger when units are on average sensitive to \mathbf{x}_t (through β_S), unit i is sensitive to spillovers (through ϕ_i), and when units are on average sensitive to spillovers (through ϕ_S).

The spillover coefficients in Equation 5 can be estimated by using RGIV after residualizing the outcome variable r_{it} with respect to observed variables \mathbf{x}_t . Residualizing purges r_{it} of the variation induced by the direct and indirect effects of \mathbf{x}_t on r_{it} . Concretely, the procedure has two steps:

1. Compute the residual \hat{r}_{it} of the regression of r_{it} on \mathbf{x}_t .
2. Treating \hat{r}_{it} as data, estimate ϕ using the RGIV estimator described in Definition 1.

Conveniently, estimation uncertainty of the first step’s regression coefficients has no effect on the asymptotic variance of the RGIV estimator in the second step. Thus, treating \hat{r}_{it} as data in the second step produces valid standard errors for $\hat{\phi}^{RGIV}$. See Appendix B.1 for formal results.

5.2 Non-identification when explanatory variables are unobserved

In this section, I describe the tradeoff in assumptions between allowing spillover coefficient heterogeneity/unknown shock variances and a factor structure for the shocks. I show that global identification of unit-specific spillover coefficients is lost when a single unobserved factor is included in the error term. Such a case is empirically relevant because there is typically no a priori reason to believe that spillover coefficients are homogeneous across units and because estimated latent factors are commonly used as control variables in the applied GIV literature.

GIV regressions with estimated latent factors as control variables are ubiquitous in the applied GIV literature (Flynn and Sastry (2022); Gabaix and Koijen (2020, 2021); Camanho et al.

(2022); Baumeister and Hamilton (2023); Adrian et al. (2022) among others). Typically, latent factors are estimated using principal components on the demeaned outcome variable before being included as control variables in the GIV regression. Such an approach is attractive because it enables practitioners to apply GIV to applications where the correlations between unit shocks are driven by a small number of latent factors. These estimated factors however are subject to measurement error, so their inclusion as control variables in subsequent GIV regressions give rise to attenuation bias.

Banafti and Lee (2022) address this concern by extending GIV with homogeneous spillover coefficients to a large time and panel dimension framework. When the size distribution of units is very skewed (more skewed than Zipf's law), the sampling uncertainty arising from latent factors and loadings is negligible under their procedure. Homogeneity of spillover coefficients across units is crucial for the procedure's validity. When spillover coefficients are heterogeneous, cross-sectionally demeaning each unit no longer differences away the (no longer constant) contribution of spillovers. As a result, PCA estimates of the latent factors are polluted by the presence of spillovers.

Given the empirical relevance of unit-level heterogeneity and latent factors, I consider a heterogeneous spillover coefficient latent factor model. Taking n to be fixed, restrictions on the skewness of the unit size distribution are unneeded. Investigating identification, I augment Assumption 1(i) to include a single latent factor f_t with unknown unit-specific loading λ_i

$$r_{it} = \phi_i r_{st} + \lambda_i f_t + u_{it}, \quad \lambda_i \neq 0, \quad f_t \perp\!\!\!\perp u_{it}, \quad n \geq 5 \quad (6)$$

where the number of units n is fixed and time $T \rightarrow \infty$. Factor f_t is normalized so that $\mathbb{E}(f_t) = 0$, $\mathbb{E}(f_t^2) = 1$, and $\lambda_1 > 0$. Shocks u_{it} are idiosyncratic in that they are independent of latent factor f_t and $u_{it} \perp\!\!\!\perp u_{jt}$ for $i \neq j$. Then, extending the logic of the RGIV estimator to the single factor case, the moment function between units $i \neq j$ is

$$g_{ij}^{\text{factor}}(\mathbf{r}_t, \boldsymbol{\theta}) = (r_{it} - \phi_i r_{st})(r_{jt} - \phi_j r_{st}) - \lambda_i \lambda_j$$

where $\boldsymbol{\theta} = [\boldsymbol{\phi}', \boldsymbol{\lambda}']'$ for $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_n]'$. $g_{ij}^{\text{factor}}(\mathbf{r}_t, \boldsymbol{\theta})$ can then be stored in moment vector $g^{\text{factor}}(\mathbf{r}_t, \boldsymbol{\theta})$. When $n \geq 5$, the number of moments (a total of $n(n-1)/2$) is greater than or equal to the number of parameters to be estimated (a total of $2n$).

Lemma 2 (below) however shows that the population moment condition $g_0^{\text{factor}}(\boldsymbol{\theta}) = \mathbb{E}[g(\mathbf{r}, \boldsymbol{\theta})]$ has multiple roots, so the model described in Equation 6 is not identified.

Lemma 2. *Consider Equation 6 and let $\boldsymbol{\theta}_0$ be the true parameter. There exists $\tilde{\boldsymbol{\theta}} \neq \boldsymbol{\theta}_0$ such*

that $g_0^{\text{factor}}(\tilde{\boldsymbol{\theta}}) = 0$.

Proof. See Appendix A.5. □

The failure in the global identification condition comes from the assumptions of unknown shock variances and unknown factor loadings. If instead shock variances and factor loadings were taken to be known—mirroring Assumption 1 of Gabaix and Koijen (2020) and as is the case for the proof of Lemma 1—then the unit-specific spillover coefficients are identified.⁵ In the proof of Lemma 2, I show that factor loadings can compensate for incorrect guesses for the spillover coefficient. To see this, let $\tilde{\boldsymbol{\theta}} = [\tilde{\boldsymbol{\phi}}', \tilde{\boldsymbol{\lambda}}']'$ be a candidate root to the population moment condition. In the proof, I consider the set of solutions where the first unit’s spillover coefficient and loading equal their true values ($\tilde{\phi}_1 = \phi_1$ and $\tilde{\lambda}_1 = \lambda_1$). I then show that a subset of the population moment conditions imply that

$$\tilde{\phi}_k - \phi_k = -\frac{\lambda_1(1 - \phi_S)}{\lambda_1\lambda_S + S_1\sigma_1^2}(\tilde{\lambda}_k - \lambda_k), \quad k \geq 2.$$

In words, $\tilde{\lambda}_k$ can compensate for an incorrect spillover coefficient $\tilde{\phi}_k \neq \phi_k$. Formally, the proof shows that there is at least one $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}_0$ such that $g_0^{\text{factor}}(\tilde{\boldsymbol{\theta}}) = 0$. Moreover, $\tilde{\boldsymbol{\theta}}$ need not be “close” to the true parameter, as there is no guarantee that $\max_i \tilde{\phi}_i \geq \min_i \phi_i$ or $\min_i \tilde{\phi}_i \leq \max_i \phi_i$. In this sense, $\tilde{\boldsymbol{\phi}}$ is potentially far from the true spillover coefficient $\boldsymbol{\phi}_0$.

Lemma 2 also implies a tradeoff between modeling unit-level heterogeneity and allowing for correlated shocks. Recall that Proposition 7 of Gabaix and Koijen (2020) shows that a *homogeneous* spillover coefficient is identified when shocks admit a factor structure with unknown loadings and if shock variances are homogeneous across units. In contrast, Lemma 2 shows that global identification is lost under unrestricted heterogeneity on spillover coefficients and shock variances. Taken together, these results suggest that practitioners face a tradeoff between two empirically-relevant models.

6 Application

Applying the robust granular instrumental variables (RGIV) methodology to the Euro area sovereign yield spillovers application of a working paper version of Gabaix and Koijen (2020), this section finds strong evidence of country-level heterogeneity in the spillovers of idiosyncratic shocks.

⁵In a related case, Appendix B.4 shows that differencing RGIV moments can account for factor loadings determined by unit-specific observables.

Just as in Gabaix and Koijen (2020), the sample consists of daily data on 10-year zero coupon yields from Bloomberg from September 1, 2009 to May 31, 2018 giving a total of 2283 observations. The included countries are Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Netherlands, Portugal, Slovenia, and Spain. For the yield spread of country i (relative to Germany) y_{it} , the outcome variable r_{it} is defined as $r_{it} = \frac{y_{it} - y_{it-1}}{0.01 + y_{i,t-1}}$ just as in Gabaix and Koijen (2020). Since shocks are linearly related to the outcome variable as outlined in Assumption 1(i), r_{it} is winsorized (over time) at the 0.5 and 99.5th percentiles. Following Gabaix and Koijen (2020), size is time-varying and computed as “debt-at-risk” $S_{i,t-1} = \frac{B_{i,t-1}y_{i,t-1}}{\sum_j B_{j,t-1}y_{j,t-1}}$ where $B_{i,t-1}$ is the outstanding government debt of country i .

Observed explanatory variables are included to account for unit-specific exposures to latent aggregate shocks. In particular I include the STOXX 50 Volatility Index (differences), the STOXX Europe 600 Index (growth), the EUR-USD exchange rate (growth), the United States 10 Year Treasury yield (growth), BBB/Baa-10Y spread (differences), and the European Fama-French 5 factors (Fama and French, 2015). These observed explanatory variables account for differential exposure of countries to uncertainty, exchange rates, equity prices, and risk. All data are downloaded from Bloomberg.

Even with observed explanatory variables, well-documented regional correlations among Europe’s core and periphery countries likely still drive correlations between shocks (Bayoumi and Eichengreen 1992). To address, I size-aggregate r_{it} to form larger country blocks; even if shocks *within* blocks are correlated, RGIV is still valid so long as shocks *between* blocks are uncorrelated. Hence, uncorrelatedness between blocks is a weaker condition than uncorrelatedness between countries. Specifically, I consider the following four blocks:

- Block 1 (*Core*): Austria, Belgium, Finland, France, Netherlands;
- Block 2 (*Western periphery*): Ireland, Portugal, and Spain;
- Block 3 (*Eastern periphery*): Greece and Italy;
- Block 4: Slovenia.

Block 1 includes countries typically classified as being members of the EU “core.” Blocks 2 and 3 contain countries typically classified as being members of the EU “periphery.” Block 4 contains Slovenia, which is typically left uncategorized on account of its distinct institutional structure as a former member of Yugoslavia.

I report RGIV point estimates and standard errors for individual spillover coefficients. Standard errors are computed using a HAC GMM weight matrix to account for the possible

serial correlation of idiosyncratic shocks.⁶ Size-weighted spillover coefficients are constructed using the Delta method, using the average block size over the estimation sample.

I also present GIV results based on the shock variance approximation and factor estimation procedures detailed in Section 5.3 of the July 2021 working paper version of Gabaix and Koijen (2020). I include this procedure for comparison, as it is used in the original application. This procedure approximates the variance of the shocks with $\text{Var}(r_{it})$, valid when spillovers are small. As a diagnostic for GIV instrument strength, the first stage F -statistic is also reported.

I find strong evidence of spillovers in the aggregate and of spillover heterogeneity across countries. In the preferred specification (Column 1 of Table 1), the null hypothesis of uncorrelated idiosyncratic shocks isn't rejected suggesting that the RGIV moment conditions are consistent with the data. Evidence of spillovers in the aggregate, the size-weighted spillover coefficient is 0.54 (with a standard error of 0.08). Moreover, with a p -value of < 0.001 for the spillover coefficient homogeneity test, the null hypothesis of spillover coefficient homogeneity is rejected at conventional significance levels. Turning to individual spillover coefficients, the spillover coefficient for the western periphery block (0.83) is more than twice that of the core countries (0.40).

The qualitative features of the preferred specification are robust to omitting observed explanatory variables and using an estimator that is efficient under higher-order dependence among shocks. Omitting explanatory variables, Column 2 of Table 1 shows spillover coefficient heterogeneity across blocks. Relative to the preferred specification, the estimated size-weighted spillover coefficient is slightly larger (at 0.63). Recall that the RGIV estimator is efficient under the independence of idiosyncratic shocks, but isn't guaranteed to be efficient under the weaker condition of idiosyncratic shock uncorrelatedness.⁷ To address, Column 3 shows the results of a "higher-order efficient" estimator, which uses the procedure outlined in Remark 5 of Section 3.2. The point estimates and standard errors are nearly identical, suggesting that higher order dependence of idiosyncratic shocks play a minor role in this application. The spillover coefficient estimated with GIV procedure featured in a working paper version of Gabaix and Koijen (2020) is comparable to the size-weighted spillover coefficient computed with RGIV, but is unable to speak to country-level heterogeneity. Column 4 of Table 1 presents results of the baseline GIV methodology applied to this section's core-periphery-aggregated panel (distinct from the country-level panel of Gabaix and Koijen (2020)). The spillover coefficient of 0.57 is close to the size-aggregated spillover coefficient computed in the preferred RGIV specification.

⁶Specifically, a Newey-West kernel with the Lazarus et al. (2018) truncation parameter rule of $1.3\sqrt{T}$.

⁷For example, a shared volatility term would induce dependence of higher order moments.

	Preferred	No controls	Higher-order efficient	1-factor GIV
RGIV results:				
ϕ_S	0.54 (0.08)	0.63 (0.07)	0.54 (0.08)	
$\phi_{\text{IRL,PRT,ESP}}$	0.83 (0.06)	0.86 (0.05)	0.83 (0.06)	
$\phi_{\text{GRC, ITA}}$	0.44 (0.16)	0.56 (0.13)	0.44 (0.16)	
ϕ_{Core}	0.39 (0.03)	0.47 (0.02)	0.40 (0.03)	
ϕ_{SVN}	0.33 (0.06)	0.49 (0.05)	0.33 (0.05)	
ϕ^{GIV}				0.56 (0.02)
Tests (p -values):				
Specification	0.950	0.923	0.935	
Homogeneity	<0.001	<0.001	<0.001	
First-stage F -statistic				592

Table 1: Spillover coefficient estimation results. RGIV coefficient estimates for ϕ_S , $\phi_{\text{IRL,PRT,ESP}}$, $\phi_{\text{GRC, ITA}}$, ϕ_{Core} and ϕ_{SVN} are listed above standard errors, which are provided in parentheses. p values are provided in the bottom section of the table for the specification test and parameter homogeneity tests. Estimates for the GIV spillover coefficient and standard error are listed in the ϕ^{GIV} row, and the the instrumental variables first-stage F -statistic is given in the table's last row. See the main text for details on the table's columns.

Section B.2 gives additional RGIV results under alternative winsorizations, the more widely-used Andrews (1991) truncation parameter, omitting the Fama-French factors as observed explanatory variables, and alternative choices of country blocking. Section B.2 also contains results for the 0-factor and 2-factor GIV specifications.

Summarizing, RGIV finds strong evidence of country-level differences in sovereign yield spillovers. In response to a 1% increase in the size-weighted relative yield spread, the relative yield spread of “core” countries increases by 0.4% compared to an increase of 0.8% for countries in the western “periphery.” Substantively, these estimates point to the importance of understanding the role of country-level characteristics in the heterogeneous propagation of idiosyncratic shocks during sovereign debt crises.

7 Simulation study

I close by showing that RGIV has good finite sample performance in simulation using four DGPs based on the preferred specification studied in Section 6. In contrast, I show that two versions of the GIV procedures—the “feasible” GIV procedure (featured in this paper’s application application) and “oracle” GIV procedure that takes idiosyncratic shock variances σ_i^2 to be known—can severely under-cover under spillover coefficient heterogeneity.

SETUP The “homogeneous spillovers” DGP is loosely based on the preferred specification of Section 6. Under spillover coefficient homogeneity across units, the DGP serves as a baseline, as both the oracle GIV and RGIV estimators are correctly specified. For $n = 4$ units and a sample length of $T = 2283$, the model parameters are below:

$$\begin{aligned} \boldsymbol{\phi} &= [0.54 \quad 0.54 \quad 0.54 \quad 0.54]', & \boldsymbol{\sigma} &= [0.014 \quad 0.014 \quad 0.014 \quad 0.014]' \\ \mathbf{S} &= [0.29 \quad 0.56 \quad 0.14 \quad 0.01]'. \end{aligned}$$

Having established an environment for which both the oracle GIV and RGIV estimators are valid, I study three additional DGPs that deviate from the homogeneous spillovers DGP. These illustrate the finite sample properties of RGIV and GIV. First, the “coefficient outlier” DGP takes the homogeneous spillovers specification and sets the spillover coefficient of the fourth unit to 0.75. Second, the “shock variance outlier” DGP takes the homogeneous spillovers specification and sets the shock standard deviation of the first unit to 0.03. Third, the “application” DGP allows for heterogeneous spillover coefficients and shock variances by assigning them to be the estimated values from the application’s preferred specification.

For each DGP, I consider results from the RGIV estimator, “feasible” GIV estima-

tor, and “oracle” estimator. For the RGIV estimator, I compute confidence intervals for the spillover coefficients of individual units, size-weighted spillover coefficients, and equal-weighted spillover coefficients. These confidence intervals highlight how RGIV can be used for researchers interested in individual spillover coefficients and for those interested in a single, aggregated parameter. The “feasible” GIV estimator is computed using the $\text{Var}(r_{it}) \approx \text{Var}(u_{it})$ approximation (valid when spillovers are small) while the “oracle” GIV estimator is computed as if the true idiosyncratic shock variance $\text{Var}(u_{it})$ were known. Both GIV estimators are included to show the practical implications of mistakenly assuming spillover coefficient homogeneity across units. The feasible GIV estimator, in particular, is included to evaluate the approximation $\text{Var}(r_{it}) \approx \text{Var}(u_{it})$ and for completeness (it is featured in the application section).

A conservative notion of coverage is used for the GIV estimators. Since the GIV estimators require spillover coefficient homogeneity across units, I report the proportion of GIV confidence intervals that contain *any* positive-weighted average of potentially heterogeneous spillover coefficients—measured as the proportion of GIV confidence intervals with a nonempty intersection with the closed interval $[\min_{1 \leq i \leq n} \phi_i, \max_{1 \leq i \leq n} \phi_i]$.

RESULTS While the empirical coverage is near the nominal level for RGIV, feasible RGIV can severely under-cover. Table 2 shows that nearly 95% of the confidence intervals for $\hat{\phi}_S$ and $\hat{\phi}_E$ contain the true estimands ϕ_S and ϕ_E in all four DGPs. In contrast, the true spillover coefficient is contained in none of the feasible GIV confidence intervals in the homogeneous spillovers DGP. Considering that the true coefficient is contained in 95% of the oracle estimator confidence intervals, together these results suggest that the approximation $\text{Var}(r_{it}) \approx \text{Var}(u_{it})$ is inappropriate for this DGP.

Under spillover coefficient heterogeneity, the GIV estimators can substantially under-cover—even with the conservative notion of coverage featured in this simulation study. For the coefficient outlier DGP, none of the feasible GIV confidence intervals contain *any* positive weighted average of heterogeneous spillovers, compared to 15% for the oracle estimator. Hence, for certain DGPs, mistakenly assuming spillover coefficient homogeneity and applying versions of the baseline GIV estimator can result in unreliable inference.

Evaluating the properties of the testing procedures featured in Section 4, the empirical size of the RGIV specification and coefficient homogeneity tests are near their nominal level. The empirical rejection rate of the coefficient homogeneity test is 4.2% and 5.2% for the homogeneous spillovers and variance outlier DGPs respectively, where the null hypothesis of spillover coefficient homogeneity across units is true. When spillover coefficients are not homogeneous, as is the case under the elasticity outlier and application DGPs, the null hy-

DGP	RGIV		GIV		Testing ($\alpha = 0.05$)	
	$\widehat{\phi}_S$	$\widehat{\phi}_E$	Feasible	Oracle	Spec.	Homog.
Homogeneous spillovers	0.94 (0.12)	0.97 (0.038)	0 (0.058)	0.95 (0.046)	0.054	0.042
Coefficient outlier	0.94 (0.12)	0.97 (0.037)	0 (0.078)	0.15 (0.071)	0.047	0.998
Variance outlier	0.97 (0.045)	0.94 (0.067)	0.0068 (0.037)	0.95 (0.033)	0.045	0.052
Application	0.95 (0.14)	0.97 (0.052)	1.00 (0.083)	1.00 (0.10)	0.064	0.996

Table 2: Simulation results (5000 simulations, $T = 2283$, 95% confidence intervals). Empirical coverage probabilities are listed above median confidence interval lengths, which are provided in parentheses. RGIV: Coverage probabilities are computed as the proportion of confidence intervals containing the estimand (ϕ_S for $\widehat{\phi}_S$ and ϕ_E for $\widehat{\phi}_E$). GIV: Coverage probabilities are computed as the proportion of confidence intervals containing any positive-weighted average of ϕ_i for the feasible and oracle GIV estimators. Computed at the 5% level, the RGIV specification test (Spec.) and coefficient homogeneity test (Homog.) are provided in the final two columns.

pothesis of homogeneous coefficients is rejected in more than 99% of Monte Carlo replications. Turning to the specification test, note that the RGIV framework is correctly specified in all four DGPs. At worst, the null hypothesis of idiosyncratic shock uncorrelatedness is rejected in 6.4% of Monte Carlo replications under the application DGP.

The RGIV estimator presents a modest (if any) power tradeoff relative to the oracle GIV estimator, and unit-specific RGIV spillover coefficients have good coverage properties despite applying to a more general environment. In the homogeneous spillovers application, the equal-weighted RGIV spillover coefficient has a comparable median confidence interval length (0.038) to that of GIV (0.046).⁸ Turning to unit-specific spillover coefficients, Table 3 shows that the empirical coverage across all four DGPs and RGIV spillover coefficients is near 0.95.

8 Discussion

Gabaix and Koijen (2020) introduces granularity-based identification, a substantial step forward for credible spillover estimates in macroeconomics and finance. Its baseline granular instrumental variables procedure however requires strong assumptions—namely homogeneous spillovers across units, homogeneous shock variances, skewed unit size, and idiosyncratic

⁸In general, researchers interested in *any* positive-weighted average of potentially heterogeneous spillover coefficients could estimate unit-specific spillover coefficients using RGIV and compute the weights that minimize the confidence interval length of the resulting aggregated estimator.

DGP	ϕ_1	ϕ_2	ϕ_3	ϕ_4
Homogeneous spillovers	0.96 (0.16)	0.95 (0.3)	0.95 (0.075)	0.95 (0.058)
Coefficient outlier	0.95 (0.16)	0.95 (0.3)	0.95 (0.075)	0.94 (0.058)
Variance outlier	0.95 (0.43)	0.96 (0.18)	0.95 (0.046)	0.95 (0.044)
Application	0.96 (0.12)	0.95 (0.32)	0.96 (0.091)	0.95 (0.14)

Table 3: Simulation results for the RGIV unit-specific spillover coefficients (5000 simulations, $T = 2283$, 95% confidence intervals). Empirical coverage probabilities are listed above median confidence interval lengths, which are provided in parentheses.

shocks (after accounting for common factors with known loadings). I build on this innovative approach by showing that unit-specific spillover coefficient heterogeneity with heterogeneous (and unknown) shock variances can jointly be accounted for without further restrictions. My estimator, called robust granular instrumental variables, also allows for homogeneous unit sizes unlike GIV. Intuitively, my approach uses internally estimated individual idiosyncratic shocks as instruments. I give results on identification and inference, showing that the required GIV assumption of coefficient homogeneity is directly testable. Relaxing the idiosyncratic shock assumption, I highlight an tradeoff: practitioners must choose between allowing unrestricted unit-level heterogeneity and a general shock covariance structure. Studying Euro zone sovereign yields, I find strong evidence of country-level heterogeneity in spillovers. I find that my proposed estimator has good finite sample properties through simulation.

There are several directions for future work. First, my approach builds on the baseline framework proposed by Gabaix and Koijen (2020). Here, an individual unit's outcome is determined by the size-weighted aggregate outcome. A structure where spillover responses are allowed to differ by the shock's source could be of interest for empirical work. Second, while the core-periphery structure of Euro area countries serves as a natural basis for grouping units in the empirical application, grouping could be automated. Third, as discussed in Section 5.2, practitioners face a tradeoff between unrestricted unit-level heterogeneity in spillover coefficients and correlated shocks. Further work on alternative economically-motivated conditions that preserve global identification would be of considerable interest to applied users.

A Proofs

A.1 Proof of Theorem 1

I will begin by showing that the hypotheses of Theorem 3.1 of Pakes and Pollard (1989) are satisfied by Assumption 1. Doing so gives consistency of a GMM estimator with moment function $g(\mathbf{r}_t, \boldsymbol{\phi})$ and the weight matrix taken to be the identity matrix. To do so, I proceed by verifying the (more stringent) conditions of Newey and McFadden (1994) Theorem 2.6, taking the weight matrix as the identity matrix. I close by showing that the constructed weight matrix $\widehat{W}(\boldsymbol{\phi})$ satisfies Lemma 3.4 of Pakes and Pollard (1989), implying consistency of the RGIV estimator.

Below, I verify the conditions of Theorem 2.6 of Newey and McFadden (1994) for an identity weight matrix $I_{n(n-1)/2}$ (preserving the same numbering as the original reference):

- i. Since the weight matrix $\widehat{W} = I_{n(n-1)/2}$ is taken to be the identity matrix, trivially $\widehat{W} \xrightarrow{p} I_{n(n-1)/2}$. Lemma 1 shows $\mathbb{E}[g(\mathbf{r}_t, \boldsymbol{\phi})] = 0$ if and only if $\boldsymbol{\phi} = \boldsymbol{\phi}_0$.
- ii. Parameter space $\boldsymbol{\Phi}$ is compact from Assumption 1(iii).
- iii. By inspection, $g(\mathbf{r}_t, \boldsymbol{\phi})$ is continuous at each $\boldsymbol{\phi}$ with probability 1.
- iv. Take $\tilde{\boldsymbol{\phi}} \in \boldsymbol{\Phi}$. Then, consider the element of $g(\mathbf{r}_t, \tilde{\boldsymbol{\phi}})$ corresponding to the orthogonality of shocks to units i and j . Then,

$$\begin{aligned} (r_{it} - \tilde{\phi}_i r_{St})(r_{jt} - \tilde{\phi}_j r_{St}) &= [u_{it} + (\phi_i - \tilde{\phi}_i)r_{St}][u_{jt} + (\phi_j - \tilde{\phi}_j)r_{St}] \quad (\text{Assumption 1(i)}) \\ &= [u_{it} + (\phi_i - \tilde{\phi}_i)\frac{u_{St}}{1 - \phi_S}][u_{jt} + (\phi_j - \tilde{\phi}_j)\frac{u_{St}}{1 - \phi_S}]. \end{aligned}$$

Thus, the moment condition is quadratic in idiosyncratic shocks. Since $\mathbb{E}[\|\mathbf{u}_t\|^2] < \infty$, $\mathbb{E}[\sup_{\boldsymbol{\phi} \in \boldsymbol{\Phi}} \|g(\mathbf{r}, \boldsymbol{\phi})\|] < \infty$.

Since the more stringent conditions of Newey and McFadden (1994) Theorem 2.6 are satisfied, the conditions of Theorem 3.1 of Pakes and Pollard (1989) are also satisfied.

Since $\widehat{W}(\boldsymbol{\phi})$ is positive definite, it can be decomposed as $\widehat{W}(\boldsymbol{\phi}) = \widehat{A}(\boldsymbol{\phi})'\widehat{A}(\boldsymbol{\phi})$ where $\widehat{A}(\boldsymbol{\phi}) = \sqrt{\widehat{W}(\boldsymbol{\phi})}$ where $\sqrt{\cdot}$ represents the element-wise square root operator. Next, proceed by verifying the conditions of Lemma 3.4 of Pakes and Pollard (1989) (preserving the same numbering as the original reference). Conditions (i) and (ii) follow from the law of large numbers, continuous mapping theorem, and $0 < \sigma_i^2 < \infty$. Hence, RGIV is consistent.

A.2 Proof of Theorem 2

I will first show that the hypotheses of Theorem 3.3 of Pakes and Pollard (1989) are satisfied by Assumption 1. Doing so gives asymptotic normality of a GMM estimator with moment function $g(\mathbf{r}_t, \boldsymbol{\phi})$ and the weight matrix taken to be the identity matrix. I begin by verifying the (more stringent) conditions of Newey and McFadden (1994) Theorem 3.4 where the weight matrix is taken to be the identity matrix. Then, I close by showing that the estimator with weight matrix $\widehat{W}(\boldsymbol{\phi})$ satisfies Lemma 3.5 of Pakes and Pollard (1989), implying that the RGIV estimator is asymptotically normal with asymptotic variance $(G'WG)^{-1}G'W\Sigma WG(G'WG)^{-1}$.

Below, I verify the conditions of Newey and McFadden (1994) Theorem 3.4. Before doing so, I pre-compute $\nabla_{\theta}g(\mathbf{r}_t, \tilde{\boldsymbol{\phi}})$ for $\tilde{\boldsymbol{\phi}} \in \Phi$

$$\nabla_{\theta}g(\mathbf{r}_t, \tilde{\boldsymbol{\phi}}) = \begin{bmatrix} M_2 \\ M_3 \\ \vdots \\ M_n \end{bmatrix}$$

$$M_i = \begin{bmatrix} & -r_{St}(r_{it} - \tilde{\phi}_i r_{St}) & & \\ 0_{(n-i+1) \times (i-2)} & -r_{St}(r_{(i+1)t} - \tilde{\phi}_{i+1} r_{St}) & -I_{n-i+1} \cdot r_{St}(r_{(i-1)t} - \tilde{\phi}_{i-1} r_{St}) & \\ & \vdots & & \\ & -r_{St}(r_{nt} - \tilde{\phi}_n r_{St}) & & \end{bmatrix}$$

Also, for more compact notation, let $K = \sum_{i=1}^n S_i^2 \sigma_i^2$ where σ_i^2 is given by Assumption 1(ii). Moreover, the hypotheses of Newey and McFadden (1994) Theorem 2.6 are satisfied in Theorem 1. I check the hypotheses of Theorem 3.4 of Newey and McFadden (1994) for identity weight matrix $\widehat{W} = I_{n(n-1)/2}$ (preserving the numbering of the original reference):

- i. By Assumption 1(iii), $\boldsymbol{\phi}_0 \in \text{interior}(\Phi)$.
- ii. Inspecting the functional form of $\nabla_{\theta}g(\mathbf{r}, \tilde{\boldsymbol{\phi}})$, $g(\mathbf{r}, \tilde{\boldsymbol{\phi}})$ is continuously differentiable in a neighborhood \mathcal{N} of $\boldsymbol{\phi}_0$ with probability approaching 1.
- iii. $\mathbb{E}[g(\mathbf{r}, \boldsymbol{\phi}_0)] = 0$ is shown in Lemma 1. The square of the shock orthogonality condition between countries i and j is bounded

$$\mathbb{E}[u_{it}^2 u_{jt}^2]^2 \leq \mathbb{E}[u_{it}^4] \mathbb{E}[u_{jt}^4] < \infty.$$

The Cauchy-Schwarz inequality gives the first inequality. The second inequality follows

from the bounded fourth moment condition in Assumption 1(ii). Thus $\mathbb{E}[\|g(\mathbf{r}, \boldsymbol{\phi}_0)\|^2]$ is finite.

iv. Fix $i \in 1, \dots, n$ and $\tilde{\phi} \in \boldsymbol{\Phi}$. Then

$$\begin{aligned} \|-r_{St}(r_{it} - \tilde{\phi}_i r_{St})\| &= \|-r_{St}(u_{it} + (\phi_i - \tilde{\phi}_i)r_{St})\| && \text{(Assumption 1(i))} \\ &\leq \|r_{St}u_{it}\| + \|\phi_i - \tilde{\phi}_i\| \cdot \|r_{St}^2\| \\ &= \left\| \frac{u_{St}}{1 - \phi_S} u_{it} \right\| + \|\phi_i - \tilde{\phi}_i\| \cdot \left\| \left(\frac{u_{St}}{1 - \phi_S} u_{it} \right)^2 \right\|. \end{aligned}$$

The above expression is quadratic in idiosyncratic shocks u_{it} . From Assumption 1(ii), $\mathbb{E}[\mathbf{u}_t \mathbf{u}_t']$ exists and $\mathbb{E}[\sup_{\phi \in \boldsymbol{\Phi}} \|-r_{St}(r_{it} - \tilde{\phi}_i r_{St})\|] < \infty$. Hence, $\mathbb{E}[\sup_{\phi \in \boldsymbol{\Phi}} \|\nabla_{\phi} g(\mathbf{r}, \boldsymbol{\phi})\|] < \infty$.

v. To show $G'G$ is full rank, it suffices to show that $\text{rank}(G) = n$. Recall $G = \mathbb{E}[\nabla_{\phi} g(\mathbf{r}_t, \boldsymbol{\phi}_0)]$. Since $\mathbb{E}[-r_{St}(r_{it} - \phi_i r_{St})] = \mathbb{E}[r_{St}u_{it}] = -\mathbb{E}\left[\frac{u_{St}u_{it}}{1 - \phi_S}\right] = -\frac{S_i \sigma_i^2}{1 - \phi_S}$, G can be explicitly computed:

$$G = \begin{bmatrix} \mathbb{E}(M_2) \\ \mathbb{E}(M_3) \\ \vdots \\ \mathbb{E}(M_n) \end{bmatrix}, \quad \mathbb{E}(M_i) = -\frac{1}{1 - \phi_S} \begin{bmatrix} S_i \sigma_i^2 & & & \\ S_{i+1} \sigma_{i+1}^2 & & & \\ \vdots & & & \\ S_n \sigma_n^2 & & & \\ & & I_{n-i+1} \cdot S_{i-1} \sigma_{i-1}^2 & \end{bmatrix}.$$

To show G is full rank, consider the $n \times n$ submatrix $G_{1:n, 1:n}$ formed by taking the first n rows of G . Then, its transpose is

$$G'_{1:n, 1:n} = -\frac{1}{1 - \phi_S} \begin{bmatrix} S_2 \sigma_2^2 & S_3 \sigma_3^2 & \dots & S_n \sigma_n^2 & 0 \\ & & & & S_3 \sigma_3^2 \\ & & & & S_2 \sigma_2^2 \\ & & & S_1 \sigma_1^2 I_{n-1} & 0 \\ & & & & \vdots \\ & & & & 0 \end{bmatrix}.$$

The determinant of $G'_{1:n,1:n}$ can be computed by applying elementary row operations:

$$\begin{aligned}
\det(G'_{1:n,1:n}) &= \det \left(-\frac{1}{1-\phi_S} \begin{bmatrix} S_2\sigma_2^2 & S_3\sigma_3^2 & \dots & S_n\sigma_n^2 & 0 \\ & & & & S_3\sigma_3^2 \\ & & & & S_2\sigma_2^2 \\ & S_1\sigma_1^2 I_{n-1} & & & 0 \\ & & & & \vdots \\ & & & & 0 \end{bmatrix} \right) \\
&= (-1)^{n-1} \det \left(-\frac{1}{1-\phi_S} \begin{bmatrix} & & & & S_3\sigma_3^2 \\ & & & & S_2\sigma_2^2 \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ S_2\sigma_2^2 & S_3\sigma_3^2 & \dots & S_n\sigma_n^2 & 0 \end{bmatrix} \right) \\
&= (-1)^{n-1} \det \left(-\frac{1}{1-\phi_S} \begin{bmatrix} & & & & S_3\sigma_3^2 \\ & & & & S_2\sigma_2^2 \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ S_2\sigma_2^2 & S_3\sigma_3^2 & 0 & \dots & 0 \end{bmatrix} \right) \\
&= (-1)^{n-1} \det \left(-\frac{1}{1-\phi_S} \begin{bmatrix} & & & & S_3\sigma_3^2 \\ & & & & S_2\sigma_2^2 \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ 0 & 0 & 0 & \dots & 0 -\frac{S_3\sigma_3^2}{S_1\sigma_1^2} S_2\sigma_2^2 - \frac{S_2\sigma_2^2}{S_1\sigma_1^2} S_3\sigma_3^2 \end{bmatrix} \right) \\
&= (-1)^{n-1} \left(-\frac{1}{1-\phi_S} \right)^n [S_1\sigma_1^2]^{n-1} \left[-\frac{S_3\sigma_3^2}{S_1\sigma_1^2} S_2\sigma_2^2 - \frac{S_2\sigma_2^2}{S_1\sigma_1^2} S_3\sigma_3^2 \right] \\
&= 2 \left(\frac{1}{1-\phi_S} \right)^n (S_1\sigma_1^2)^{n-2} S_2 S_3 \sigma_2^2 \sigma_3^2 \neq 0.
\end{aligned}$$

In line 2, the first row is shifted to the matrix's final row. In line 3, rows are combined so that the elements 3 to $n - 1$ of the final row are canceled. In line 4, rows are combined so that the first two elements of the final row are canceled. The determinant in line 5 is the product of the matrix's diagonal elements because the matrix is upper triangular. Since $1 - \phi_S \neq 0$, $S_i > 0$, and $\sigma_i^2 > 0$, the determinant is not zero. Hence,

submatrix $G'_{1:n,1:n}$ is full rank, which immediately implies G is also full rank:

$$n = \text{rank}(G'_{1:n,1:n}) \leq \text{rank}(G) \leq \min\left(\frac{n(n-1)}{2}, n\right) = n.$$

Since the more stringent conditions of Newey and McFadden (1994) Theorem 3.4(i)-(v) are satisfied, the conditions of Theorem 3.3 of Pakes and Pollard (1989) are also satisfied. Continue to verifying the conditions of Lemma 3.5 of Pakes and Pollard (1989). Recall from the proof of Theorem A.1 that the weight matrix can be decomposed as $\widehat{W}(\boldsymbol{\phi}) = \widehat{A}(\boldsymbol{\phi})' \widehat{A}(\boldsymbol{\phi})$ where $\widehat{A}(\boldsymbol{\phi}) = \text{diag}\left(\sqrt{\frac{1}{T} \sum_{t=1}^T g(\mathbf{r}_t, \boldsymbol{\phi})^2}\right)^{-1}$ for element-wise square root operator $\sqrt{\cdot}$. From the continuous mapping theorem and law of large numbers,

$$\widehat{A}(\boldsymbol{\phi}_0) \xrightarrow{p} \text{diag}(\sigma_1\sigma_2, \dots, \sigma_1\sigma_n, \sigma_2\sigma_3, \dots, \sigma_{n-1}\sigma_n) = A.$$

Let $\{\delta_n\}$ be a sequence of positive numbers that converges to zero. Then,

$$\sup_{\|\boldsymbol{\phi} - \boldsymbol{\phi}_0\| < \delta_n} \|\widehat{A}(\boldsymbol{\phi}) - A\| = o_p(1)$$

from continuity of the map from $\boldsymbol{\phi}$ to $\widehat{A}(\boldsymbol{\phi})$ at $\boldsymbol{\phi} = \boldsymbol{\phi}_0$. Hence, the RGIV estimator is asymptotically normal with asymptotic variance $(G'WG)^{-1}G'W\Sigma WG(G'WG)^{-1}$ for $W = A(\boldsymbol{\phi}_0)'A(\boldsymbol{\phi}_0) = \text{diag}(\sigma_1^2\sigma_2^2, \dots, \sigma_1^2\sigma_n^2, \sigma_2^2\sigma_3^2, \dots, \sigma_{n-1}^2\sigma_n^2)$.

A.3 Proof of Proposition 1

Before computing the GIV estimand, $\mathbb{E}[(u_{St} - u_{Et})u_{Et}]$, $\mathbb{E}[z_t u_{Et}]$, and $\mathbb{E}[(r_{St} - r_{Et})r_{St}]$ are precomputed below:

$$\mathbb{E}[(u_{St} - u_{Et})u_{Et}] = \frac{1}{n} \sum_{i=1}^n = \mathbb{E}\left[\left(\sum_{i=1}^n (S_i - \frac{1}{n})u_{it}\right) \frac{1}{n} \sum_{i=1}^n u_{it}\right] = \frac{1}{n} \sum_{i=1}^n (S_i - \frac{1}{n})\sigma^2 = 0.$$

$$\begin{aligned} \mathbb{E}[z_t u_{Et}] &= \mathbb{E}[(r_{St} - r_{Et})u_{Et}] = \mathbb{E}[(\phi_S - \phi_E)r_{St}u_{Et} + (u_{St} - u_{Et})u_{Et}] \\ &= (\phi_S - \phi_E)\mathbb{E}[r_{St}u_{Et}] = \frac{\phi_S - \phi_E}{1 - \phi_S} \mathbb{E}[u_{St}u_{Et}] \\ &= \frac{\phi_S - \phi_E}{1 - \phi_S} \frac{1}{n} \sigma^2. \end{aligned}$$

$$\begin{aligned} \mathbb{E}[(r_{St} - r_{Et})r_{St}] &= (\phi_S - \phi_E)\mathbb{E}[r_{St}^2] + \mathbb{E}[r_{St}(u_{St} - u_{Et})] \\ &= \frac{(\phi_S - \phi_E)\sigma^2 \sum_{i=1}^n S_i^2}{(1 - \phi_S)^2} + \frac{\sigma^2}{1 - \phi_S} \left[\sum_{i=1}^n S_i^2 - \frac{1}{n}\right]. \end{aligned}$$

Combining, the GIV estimand is

$$\begin{aligned}\frac{\mathbb{E}[z_t r_{Et}]}{\mathbb{E}[z_t r_{St}]} &= \frac{\mathbb{E}[z_t(\phi_E r_{St} + u_{Et})]}{\mathbb{E}[z_t r_{St}]} = \phi_E + \frac{\mathbb{E}[z_t u_{Et}]}{\mathbb{E}[z_t r_{St}]} \\ &= \phi_E + \frac{\phi_S - \phi_E}{n} \frac{1}{\frac{\phi_S - \phi_E}{1 - \phi_S} \sum_{i=1}^n S_i^2 - \frac{1}{n} + \sum_{i=1}^n S_i^2}\end{aligned}$$

The first line follows from Assumption 1(i).

A.4 Proof of Lemma 1

I prove Lemma 1 under the weaker condition that the covariance of shocks i and j are known. Explicitly, the below proof will use Assumption 1 after replacing Condition (ii) with Condition (ii'):

Condition (ii'). For $\sigma_i^2 > 0$, shocks \mathbf{u}_t are i.i.d. with moments $\mathbb{E}(\mathbf{u}_t) = 0$, $\mathbb{E}(u_{it}^2) = \sigma_i^2$, and $\mathbb{E}(\|\mathbf{u}_t\|^4) < \infty$. Moreover, $\mathbb{E}(u_{it}u_{jt}) = \mu_{ij}$ for known μ_{ij} .

Take $\tilde{\phi} \in \Phi$. The moment condition corresponding to the idiosyncratic shocks of units i and j

$$\begin{aligned}\tilde{\mu}_{ij} &= \mathbb{E}[(r_{it} - \tilde{\phi}_i r_{St})(r_{jt} - \tilde{\phi}_j r_{St})] \\ &= \mathbb{E}[(\phi_i - \tilde{\phi}_i)r_{St} + u_{it})(\phi_j - \tilde{\phi}_j)r_{St} + u_{jt}] \quad (\text{Assumption 1(i)}) \\ &= (\phi_i - \tilde{\phi}_i)\mathbb{E}[r_{St}u_{jt}] + (\phi_j - \tilde{\phi}_j)\mathbb{E}[r_{St}u_{it}] + (\phi_i - \tilde{\phi}_i)(\phi_j - \tilde{\phi}_j)\mathbb{E}[r_{St}^2] + \mu_{ij}. \quad (7)\end{aligned}$$

Rearranging, the above display implies

$$\phi_i - \tilde{\phi}_i = -\frac{(\phi_j - \tilde{\phi}_j)\mathbb{E}[r_{St}u_{it}]}{\mathbb{E}[r_{St}u_{jt}] + (\phi_j - \tilde{\phi}_j)\mathbb{E}[r_{St}^2]} \quad \text{and} \quad \phi_k - \tilde{\phi}_k = -\frac{(\phi_1 - \tilde{\phi}_1)\mathbb{E}[r_{St}u_{kt}]}{\mathbb{E}[r_{St}u_{1t}] + (\phi_1 - \tilde{\phi}_1)\mathbb{E}[r_{St}^2]} \quad (8)$$

for $k > 1$. Substituting the above display into Equation 7 and rearranging gives

$$0 = \frac{(\phi_1 - \tilde{\phi}_1)\mathbb{E}[r_{St}u_{it}]\mathbb{E}[r_{St}u_{jt}]}{\mathbb{E}[r_{St}u_{1t}] + (\phi_1 - \tilde{\phi}_1)\mathbb{E}[r_{St}^2]} \left\{ -2 + \frac{(\phi_1 - \tilde{\phi}_1)\mathbb{E}[r_{St}^2]}{\mathbb{E}[r_{St}u_{1t}] + (\phi_1 - \tilde{\phi}_1)\mathbb{E}[r_{St}^2]} \right\}. \quad (9)$$

Equation 9 equals zero when either the first or second terms equal zero. Beginning with the first case, the first multiplicative term equals zero when $\tilde{\phi}_1 = \phi_1$. Applying Equation 8, $\tilde{\phi}_k = \phi_k$. Thus, the first case corresponds to the true solution $\tilde{\phi} = \phi$.

Focusing on the second case, the second multiplicative term of Equation 9 equals zero when

$$0 = -2 + \frac{(\phi_1 - \tilde{\phi}_1)\mathbb{E}[r_{St}^2]}{\mathbb{E}[r_{St}u_{1t}] + (\phi_1 - \tilde{\phi}_1)\mathbb{E}[r_{St}^2]}.$$

Rearranging terms and substituting into Equation 8 implies

$$\tilde{\phi}_S = \phi_1 - \tilde{\phi}_1 = -2\frac{\mathbb{E}[r_{St}u_{1t}]}{\mathbb{E}[r_{St}^2]} \text{ and } \phi_k - \tilde{\phi}_k = -2\frac{\mathbb{E}[r_{St}u_{kt}]}{\mathbb{E}[r_{St}^2]} \text{ for } k > 1.$$

After taking a size-weighted sum of $\tilde{\phi}_k$,

$$\sum_{i=1}^n S_i \tilde{\phi}_i = \phi_S + 2(1 - \phi_S) = 1 + (1 - \phi_S) > 1.$$

The final inequality follows from the $\phi_S < 1$ condition from Assumption 1(i). In words, the second multiplicative term of Equation 9 equals zero for a value of $\tilde{\phi}$ that is outside of the parameter space.

Hence, the only solution to $\mathbb{E}[g(\mathbf{r}_t, \tilde{\phi})] = 0$ is $\phi = \phi_0$.

A.5 Proof of Lemma 2

Let the true parameter values be given by $\boldsymbol{\theta} = [\boldsymbol{\phi}', \boldsymbol{\lambda}']'$. Will show that the system of equations given by the moment conditions doesn't admit a unique solution. That is, it need not be true that $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$.

The size-weighted outcome variable can be written as idiosyncratic shocks and and the common factor

$$r_{St} = \phi_S r_{St} + \lambda_S f_t + u_{St} = \frac{1}{1 - \phi_S} [\lambda_S f_t + u_{St}].$$

For units $i \neq j$, the moment condition can be written as

$$\begin{aligned} \tilde{\lambda}_i \tilde{\lambda}_j &= \mathbb{E}\{[r_{it} - \tilde{\phi}_i r_{St}][r_{jt} - \tilde{\phi}_j r_{St}]\} \\ &= \mathbb{E}\{[(\phi_i - \tilde{\phi}_i)r_{St} + \lambda_i f_t + u_{it}][(\phi_j - \tilde{\phi}_j)r_{St} + \lambda_j f_t + u_{jt}]\} \\ &= (\phi_i - \tilde{\phi}_i)(\phi_j - \tilde{\phi}_j) \left[\frac{\lambda_S^2 + \sum_{k=1}^n S_k^2 \sigma_k^2}{1 - \phi_S} \right]^2 + (\phi_i - \tilde{\phi}_i)\lambda_j \frac{\lambda_S}{1 - \phi_S} \\ &\quad + (\phi_j - \tilde{\phi}_j)\lambda_i \frac{\lambda_S}{1 - \phi_S} + (\phi_i - \tilde{\phi}_i) \frac{S_j \sigma_j^2}{1 - \phi_S} + (\phi_j - \tilde{\phi}_j) \frac{S_i \sigma_i^2}{1 - \phi_S} + \lambda_i \lambda_j \end{aligned} \quad (10)$$

I substitute r_{it} for the true DGP in the second line. Clearly, one solution to the above system is $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}$.

Next, guess that another solution satisfies the restrictions $\tilde{\phi}_1 = \phi_1$ and $\tilde{\lambda}_1 = \lambda_1$. Motivated by this guess, ϕ_k for $k > 1$ can be expressed in terms of ϕ_1

$$\phi_k - \tilde{\phi}_k = \frac{\tilde{\lambda}_k \tilde{\lambda}_1 - \lambda_k \lambda_1 - (\phi_1 - \tilde{\phi}_1) \left[\frac{\lambda_k \lambda_S + S_k \sigma_k^2}{1 - \phi_S} \right]}{\frac{\phi_1 - \tilde{\phi}_1}{(1 - \phi_S)^2} K + \frac{\lambda_1 \lambda_S + S_1 \sigma_1^2}{1 - \phi_S}} = \lambda_1 (1 - \phi_S) \frac{\tilde{\lambda}_k - \lambda_k}{\lambda_1 \lambda_S + S_1 \sigma_1^2}. \quad (11)$$

The final equality uses the restrictions $\tilde{\phi}_1 = \phi_1$ and $\tilde{\lambda}_1 = \lambda_1$. Then, substituting Equation 11 into Equation 10 gives

$$\tilde{\lambda}_i \tilde{\lambda}_j = K_0 (\tilde{\lambda}_i - \lambda_i) (\tilde{\lambda}_j - \lambda_j) + K_j (\tilde{\lambda}_i - \lambda_i) + K_i (\tilde{\lambda}_j - \lambda_j) + \lambda_i \lambda_j, \quad (12)$$

storing constants as $K_j = \frac{\lambda_1 (\lambda_j \lambda_S + S_j \sigma_j^2)}{\lambda_1 \lambda_S + S_1 \sigma_1^2}$ for $j > 1$ and $K_0 = \frac{\lambda_1^2 K}{(\lambda_1 \lambda_S + S_1 \sigma_1^2)^2}$.

Consider the guess $\tilde{\lambda}_j = \frac{\lambda_j - 2K_j + K_0 \lambda_j}{K_0 - 1}$. Continue by verifying that the guess satisfies Equation 12. Expanding, the left-hand side can be written as

$$LHS = \tilde{\lambda}_i \tilde{\lambda}_j = \frac{2K_0 \lambda_i \lambda_j - 2K_0 K_j \lambda_i - 2K_0 K_i \lambda_j + 4K_i K_j - 2K_j \lambda_i - 2K_i \lambda_j + \lambda_i \lambda_j K_0^2 + \lambda_i \lambda_j}{(K_0 - 1)^2}.$$

Noting that $\tilde{\lambda}_j - \lambda_j = 2\frac{\lambda_j - K_j}{K_0 - 1}$, the RHS of Equation 12 can be written as

$$\begin{aligned} RHS &= \frac{1}{(K_0 - 1)^2} \{4K_0 (\lambda_i - K_i) (\lambda_j - K_j) + 2K_j (\lambda_i - K_i) (K_0 - 1) \\ &\quad + 2K_i (\lambda_j - K_j) (K_0 - 1) + \lambda_i \lambda_j (K_0 - 1)^2\} \\ &= \frac{1}{(K_0 - 1)^2} \{2K_0 \lambda_i \lambda_j - 2K_0 K_j \lambda_i - 2K_0 K_i \lambda_j + 4K_i K_j - 2K_j \lambda_i - 2K_i \lambda_j \\ &\quad + \lambda_i \lambda_j K_0^2 + \lambda_i \lambda_j\} = LHS. \end{aligned}$$

B Additional results

B.1 RGIV with observed explanatory variables

This section contains formal results for the RGIV model with observed explanatory variables.

Assumption 2 (below) establishes conditions necessary for identification, consistency, and asymptotic normality. Assumption 2 extends Assumption 1 to include control variables \mathbf{x}_t that are independent of idiosyncratic shocks u_{it} . In the derivations that follow, it will be convenient to reparametrize the coefficient on the control variables as $\boldsymbol{\psi}_i = \boldsymbol{\beta}_i + \frac{\phi_i}{1 - \phi_S} \boldsymbol{\beta}_S$.

Then, store the re-parametrized coefficients in vector $\boldsymbol{\psi} = [\boldsymbol{\psi}'_1, \dots, \boldsymbol{\psi}'_n]'$ in vector $\boldsymbol{\theta} = [\boldsymbol{\phi}', \boldsymbol{\psi}']'$ and its corresponding parameter space as Θ .

Assumption 2. (*Baseline model*)

(i) **Model:** For $n \geq 3$ units, let fixed sizes $S_i \in (0, 1)$ sum to 1. Outcome $\mathbf{r}_t = [r_{1t}, \dots, r_{nt}]'$ responds to the size-aggregated outcome r_{St} according to spillover coefficient ϕ_i , coefficients $\boldsymbol{\beta}_i$ ($k \times 1$), observed control variables \mathbf{x}_t ($k \times 1$), and unobserved shocks $\mathbf{u}_t = [u_{1t}, u_{2t}, \dots, u_{nt}]'$

$$r_{it} = \phi_i r_{St} + \boldsymbol{\beta}'_i \mathbf{x}_t + u_{it}, \quad \forall i = 1, \dots, n, \quad \phi_S < 1.$$

(ii) **Moments:** For $\sigma_i^2 > 0$, shocks \mathbf{u}_t are i.i.d. with moments $\mathbb{E}(\mathbf{u}_t) = 0$, $\mathbb{E}(\mathbf{u}_t \mathbf{u}'_t) = \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$, $\mathbb{E}(\|\mathbf{u}_t\|^4) < \infty$. Moreover, $u_{it} \perp\!\!\!\perp u_{jt}$ for $i \neq j$. $\mathbb{E}(\mathbf{x}_t \mathbf{x}'_t) = \Sigma_{XX}$ for positive definite Σ_{XX} and $\mathbf{x}_t \perp\!\!\!\perp u_{it}$.

(iii) **Parameter space:** For spillover coefficient $\boldsymbol{\phi} = [\phi_1, \dots, \phi_n]'$, store parameters in $\check{\boldsymbol{\theta}} = [\boldsymbol{\phi}', \boldsymbol{\beta}'_1, \dots, \boldsymbol{\beta}'_n]'$. Then, the true parameter $\check{\boldsymbol{\theta}}_0$ is in the interior of parameter space $\check{\Theta}$. $\check{\Theta}$ is compact and for any $\check{\boldsymbol{\theta}} \in \check{\Theta}$, $\check{\phi}_S < 1$.

Definition 2 (below) extends the RGIV estimator to include observed explanatory variables. Step 1 estimates the total effect of control variables \mathbf{x}_t on the outcome r_{it} . Step 2 writes the moment uncorrelatedness condition after subtracting the variation induced by the observed explanatory variables. See Section 5.1 for intuition.

Definition 2. For outcome variable $\mathbf{r}_t = [r_{1t}, \dots, r_{nt}]'$, store data $\mathbf{z}_t = [\mathbf{r}'_t, \mathbf{x}'_t]'$. Let $u_i(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi}) = (r_{it} - \boldsymbol{\psi}'_i \mathbf{x}_t) - \phi_i (r_{St} - \boldsymbol{\psi}'_S \mathbf{x}_t)$ for $i = 1, \dots, n$. The moment function $g^c(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi})$ is

$$g^c(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi}) = [u_1(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi})u_2(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi}), \dots, u_1(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi})u_n(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi}), \\ u_2(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi})u_3(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi}), \dots, u_{n-1}(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi})u_n(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi})]'$$

Let the sample weight matrix \widehat{W} be defined as

$$\widehat{W} = \text{diag}\left(\frac{1}{\widehat{\sigma}_1^2 \widehat{\sigma}_2^2}, \dots, \frac{1}{\widehat{\sigma}_1^2 \widehat{\sigma}_n^2}, \frac{1}{\widehat{\sigma}_2^2 \widehat{\sigma}_3^2}, \dots, \frac{1}{\widehat{\sigma}_{n-1}^2 \widehat{\sigma}_n^2}\right)$$

where $\widehat{\sigma}_i^2$ is a consistent estimator of idiosyncratic shock variances σ_i^2 for $i = 1, \dots, n$. Then the **robust granular instrumental variables estimator with observed explanatory variables** is characterized as the following two step estimator:

1. For each $i = 1, \dots, n$, $\widehat{\boldsymbol{\psi}}_i^{\text{Step 1}} = (\frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \mathbf{x}'_t)^{-1} (\frac{1}{T} \sum_{t=1}^T \mathbf{x}_t r_{it})$.

$$2. \text{ For } \widehat{Q}_T(\boldsymbol{\phi}; \widehat{\boldsymbol{\psi}}^{\text{Step 1}}) = \left[\frac{1}{T} \sum_{t=1}^T g^c(\mathbf{z}_t, \boldsymbol{\phi}; \widehat{\boldsymbol{\psi}}^{\text{Step 1}}) \right]' \widehat{W} \left[\frac{1}{T} \sum_{t=1}^T g^c(\mathbf{z}_t, \boldsymbol{\phi}; \widehat{\boldsymbol{\psi}}^{\text{Step 1}}) \right], \widehat{\boldsymbol{\phi}}^{\text{RGIV},c} = \arg \min_{\boldsymbol{\phi}: \phi_S < 1} \widehat{Q}_T(\boldsymbol{\phi}; \widehat{\boldsymbol{\psi}}^{\text{Step 1}}).$$

For brevity, the formal results rely on straightforward extensions of the Newey and McFadden (1994) GMM identification, consistency, and asymptotic normality results as detailed in Appendix B.3 to allow for the first step estimation of a nuisance parameter. Note that the two-step estimation results of Newey and McFadden (1994) (Section 6) cannot be directly applied since there are potentially more moments than estimated parameters. Alternatively, the estimator described in Definition 2 could be defined using a continuously updating GMM objective function in the second step. The procedure could be formalized by modifying the more general results of Pakes and Pollard (1989) to accommodate a first-step estimator.

Lemma 3 (below) establishes that the first step estimator $\widehat{\boldsymbol{\psi}}^{\text{Step 1}}$ is consistent and $O_p(1)$. Lemma 4 establishes identification of spillover coefficients $\boldsymbol{\phi}$. Theorems 3 and 4 give consistency and asymptotic normality of the RGIV with observed explanatory variables estimator $\widehat{\boldsymbol{\phi}}^{\text{RGIV},c}$.

Lemma 3. *Impose Assumption 2. As $T \rightarrow \infty$, $\widehat{\boldsymbol{\psi}}_i^{\text{Step 1}} \xrightarrow{p} \boldsymbol{\psi}_i$ and $\sqrt{T}(\widehat{\boldsymbol{\psi}}_i^{\text{Step 1}} - \boldsymbol{\psi}_i) = O_p(1)$.*

Proof. Since $0 = \mathbb{E}[\mathbf{x}_t(r_{it} - \boldsymbol{\psi}'_i \mathbf{x}_t)] = \mathbb{E}[\mathbf{x}_t u_{it}] = 0$ and $\mathbb{E}[\mathbf{x}_t \mathbf{x}'_t]$ is full rank, the OLS estimator $\widehat{\boldsymbol{\psi}}_i^{\text{Step 1}}$ is consistent. Assumption 2(ii) (that $\mathbb{E}(\mathbf{u}_t \mathbf{u}'_t) = \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$ and $\mathbb{E}(\mathbf{x}_t \mathbf{x}'_t)$ full rank) implies $\widehat{\boldsymbol{\psi}}_i^{\text{Step 1}}$ is asymptotically normal so $\sqrt{T}(\widehat{\boldsymbol{\psi}}_i^{\text{Step 1}} - \boldsymbol{\psi}_i) = O_p(1)$. \square

Lemma 4 (Identification with observed explanatory variables). *Impose Assumption 2. For $g_0^c(\boldsymbol{\phi}) = \mathbb{E}[g^c(\mathbf{z}_t, \boldsymbol{\phi}; \boldsymbol{\psi}_0)]$, $g_0^c(\boldsymbol{\phi}_0) = 0$ for the true parameter $\boldsymbol{\phi}_0$ and $g_0^c(\tilde{\boldsymbol{\phi}}) \neq 0$ for $\tilde{\boldsymbol{\phi}} \in \Phi$ such that $\tilde{\boldsymbol{\phi}} \neq \boldsymbol{\phi}_0$.*

Proof. The result immediately follows after applying Lemma 1 to $\dot{r}_{it} = r_{it} - \boldsymbol{\psi}'_i \mathbf{x}_t$. \square

Theorem 3 (Consistency of RGIV with observed explanatory variables). *Impose Assumption 2. The RGIV estimator with observed explanatory variables is consistent $\widehat{\boldsymbol{\phi}}^{\text{RGIV},c} \xrightarrow{p} \boldsymbol{\phi}_0$ for the true parameter $\boldsymbol{\phi}_0$ as $T \rightarrow \infty$.*

Proof. Verify the conditions of Theorem 5. Condition (i) follows from Lemma 4 and Lemma 5. Condition (ii) follows from Assumption 2(iii). Condition (iii) follows by inspection of function $g^c(\mathbf{z}_t, \boldsymbol{\phi}, \boldsymbol{\gamma})$. Condition (iv) follows from Assumption 2(ii). \square

Theorem 4. *Impose Assumption 1. The RGIV estimator with observed explanatory variables is asymptotically normal*

$$\sqrt{T}(\widehat{\boldsymbol{\phi}}^{\text{RGIV},c} - \boldsymbol{\phi}_0) \xrightarrow{d} \mathcal{N}(0, (G'WG)^{-1}G'W\Sigma WG(G'WG)^{-1})$$

for RGIV population weight matrix $W = \text{diag}(\frac{1}{\sigma_1^2 \sigma_2^2}, \dots, \frac{1}{\sigma_1^2 \sigma_n^2}, \frac{1}{\sigma_2^2 \sigma_3^2}, \dots, \frac{1}{\sigma_{n-1}^2 \sigma_n^2})$, moment covariance matrix $\Sigma = \mathbb{E}[g^c(\mathbf{r}_t, \boldsymbol{\phi}_0; \boldsymbol{\psi}_0)g^c(\mathbf{r}_t, \boldsymbol{\phi}_0; \boldsymbol{\psi}_0)']$ and $G = \mathbb{E}[\nabla_{\boldsymbol{\phi}} g^c(\mathbf{r}_t, \boldsymbol{\phi}_0; \boldsymbol{\psi}_0)]$ as $T \rightarrow \infty$.

Proof. Verify the conditions of Theorem 6. The hypotheses for Theorem 5 are satisfied. Lemma 3 implies $\sqrt{T}(\widehat{\boldsymbol{\psi}}^{\text{Step 1}} - \boldsymbol{\psi}_0) = O_p(1)$. Next, compute the elements of $\mathbb{E}[\nabla_{\boldsymbol{\psi}} g^c(\mathbf{z}_t, \boldsymbol{\phi}_0; \boldsymbol{\psi}_0)]$. Let $i, j, k \in \{1, \dots, n\}$ and $i \neq j \neq k$. Since $\mathbf{x}_t \perp\!\!\!\perp u_{it}$,

$$\begin{aligned} & \mathbb{E}\left\{\frac{\partial}{\partial \psi_j}[r_{it} - \boldsymbol{\psi}'_i \mathbf{x}_t - \phi_i(r_{St} - \boldsymbol{\psi}'_S \mathbf{x}_t)][r_{jt} - \boldsymbol{\psi}'_j \mathbf{x}_t - \phi_j(r_{St} - \boldsymbol{\psi}'_S \mathbf{x}_t)]\right\} \\ & \quad = \mathbb{E}\{(-\mathbf{x}_t + \phi_i S_i \mathbf{x}_t)u_{jt}(\mathbf{z}_t, \boldsymbol{\psi}_0, \boldsymbol{\phi}_0)\} + \mathbb{E}[\phi_j S_i \mathbf{x}_t u_{it}(\mathbf{z}_t, \boldsymbol{\psi}_0, \boldsymbol{\phi}_0)] = 0 \\ & \mathbb{E}\left\{\frac{\partial}{\partial \psi_k}[r_{it} - \boldsymbol{\psi}'_i \mathbf{x}_t - \phi_i(r_{St} - \boldsymbol{\psi}'_S \mathbf{x}_t)][r_{jt} - \boldsymbol{\psi}'_j \mathbf{x}_t - \phi_j(r_{St} - \boldsymbol{\psi}'_S \mathbf{x}_t)]\right\} \\ & \quad = \mathbb{E}\{\phi_i S_k \mathbf{x}_t u_{jt}(\mathbf{z}_t, \boldsymbol{\psi}_0, \boldsymbol{\phi}_0)\} + \mathbb{E}[\phi_j S_k \mathbf{x}_t u_{it}(\mathbf{z}_t, \boldsymbol{\psi}_0, \boldsymbol{\phi}_0)] = 0. \end{aligned}$$

Hence, $\mathbb{E}[\nabla_{\boldsymbol{\psi}} g^c(\mathbf{z}_t, \boldsymbol{\phi}_0; \boldsymbol{\psi}_0)] = \mathbf{0}$.

Condition (i) holds from Assumption 2(iii). Condition (ii) holds by inspection of moment function $g(\mathbf{z}_t, \boldsymbol{\theta}; \boldsymbol{\gamma})$. Conditions (iii) and (iv) hold from Assumption 2(ii). Condition (v) holds from applying the proof of Theorem 2. \square

B.2 Application: Robustness

Table 4 reports coefficients and standard errors under alternative winsorization schemes. Columns 1-3 show that the results are qualitatively similar to the preferred specification after winsorizing at 5%, 0.5%, and 10% respectively. The fourth column however shows that estimation without winsorization is unreliable. These results show that the qualitative features of the main text's preferred specification aren't specific to the choice of 1% winsorization and that the linear form of the model described in Assumption 1(i) is sensitive to extreme outliers.

Table 5 summarizes RGIV results under alternative blocking schemes. Starting with the preferred specification, the first column shifts Ireland from the Portugal/Spain periphery block to the Greece/Italy periphery block (splitting the periphery countries into an "Iberia" block and "Other" block). The specification test's p -value of 0.95 suggests that "Iberia" specification's moments are consistent with the data. Moreover, the spillover coefficient for the Portugal/Spain block is 0.89 compared to 0.83 for the Portugal/Spain/Ireland block in the preferred specification, suggesting that Ireland's spillover coefficient is lower than the size-weighted combination of Portugal/Spain's. Columns 2 and 3 show that spillover coefficients are nearly unchanged after moving Slovenia to the core and periphery block respectively (no specification test is reported since the model is just-identified). The specification in Columns

	5%	0.5%	10%	0%
RGIV results:				
ϕ_S	0.57 (0.06)	0.54 (0.08)	0.58 (0.05)	-16.57 (10275.21)
$\phi_{\text{IRL,PRT,ESP}}$	0.82 (0.05)	0.82 (0.06)	0.83 (0.05)	0.98 (2.38)
$\phi_{\text{GRC, ITA}}$	0.49 (0.12)	0.45 (0.16)	0.5 (0.11)	-28.08 (17084.06)
ϕ_{Core}	0.42 (0.03)	0.38 (0.03)	0.42 (0.03)	0.47 (2.48)
ϕ_{SVN}	0.33 (0.05)	0.33 (0.06)	0.34 (0.05)	0.34 (0.39)
Tests (p -values):				
Specification	0.817	0.872	0.679	0.616
Homogeneity	0	0	0	0.275

Table 4: Estimation results. Coefficient estimates for ϕ_S , $\phi_{\text{IRL,PRT,ESP}}$, $\phi_{\text{GRC, ITA}}$, ϕ_{Core} and ϕ_{SVN} are listed above standard errors, which are provided in parentheses. p values are provided in the bottom section of the table for the specification test and parameter homogeneity tests. Columns 1-4 give estimation results for winsorization at the 2.5 and 97.5 percentiles, 0.25 and 99.75 percentiles, 5 and 95 percentiles, and no winsorization respectively.

4 separates France from the core countries and moves it into its own group. The specification test is rejected, giving indirect evidence that French shocks are correlated with those of other Euro area core countries.

B.3 Observed explanatory variables: Second-step GMM estimation with a consistent first-step estimator of a nuisance parameter

For completeness, the below identification, consistency, and asymptotic normality results are straightforward adaptations of those found in Newey and McFadden (1994) (abbreviated as NM). These results give conditions for identification and inference for two-step estimators, given a consistent first-step estimator of a nuisance parameter. Following the notation of NM, the first step nuisance parameter estimator is consistent $\hat{\gamma} \xrightarrow{p} \gamma_0$ where $\sqrt{n}(\hat{\gamma} - \gamma_0) = O_p(1)$. In the second step, θ is the parameter of interest and is estimated using GMM with moment function $g(\mathbf{z}, \theta; \gamma)$ and weight matrix estimator $\widehat{W} \xrightarrow{p} W$, where $\hat{\gamma}$ is an estimator for γ . Moreover, the condition $\mathbb{E}[\nabla_{\gamma} g(\mathbf{z}, \theta_0; \gamma_0)] = 0$ (for true parameter values γ_0 and θ_0) ensures that estimation uncertainty for the first-step estimator doesn't enter the expression for the asymptotic variance of the second-step estimator, as is the case for the RGIV estimator with

	Iberia	Core SVN	Periphery SVN	Separate FRA
RGIV results:				
ϕ_S	0.51 (0.11)	0.54 (0.08)	0.54 (0.08)	0.52 (0.09)
ϕ_I	0.89 (0.06)	0.82 (0.06)	0.83 (0.06)	0.83 (0.06)
ϕ_{II}	0.39 (0.2)	0.45 (0.16)	0.44 (0.16)	0.42 (0.18)
ϕ_{III}	0.4 (0.04)	0.39 (0.03)	0.39 (0.03)	0.4 (0.03)
ϕ_{IV}	0.33 (0.06)			0.38 (0.04)
Tests (p -values):				
Specification	0.95	–	–	0.001
Homogeneity	<.001	<.001	<.001	<.001

Table 5: Estimation results for alternative groupings. Coefficient estimates for $\phi_S, \phi_I, \phi_{II}, \phi_{III}$ and ϕ_{IV} are listed above standard errors, which are provided in parentheses. p values are provided in the bottom section of the table for the specification test and parameter homogeneity tests. Groups: “Iberia” (I: PRT, ESP; II: GRC, ITA, IRL; III: AUT, BEL, FRA, FIN, NLD; IV: SVN), “Core SVN” (I: PRT, ESP; II: GRC, ITA, IRL; III: AUT, BEL, FRA, FIN, NLD, SVN), “Periphery SVN” (I: PRT, ESP; IRL II: GRC, ITA, SVN; III: AUT, BEL, FRA, FIN, NLD; IV: SVN), “Separate FRA” (I: PRT, ESP, IRL; II: GRC, ITA; III: AUT, BEL, FIN, NLD, SVN; IV: FRA).

	Andrews rule	No Fama-French	0-factor GIV	2-factor GIV
RGIV results:				
ϕ_S	0.54 (0.07)	0.54 (0.08)		
$\phi_{PRT,ESP,IRL}$	0.83 (0.05)	0.83 (0.06)		
$\phi_{GRC,ITA}$	0.44 (0.14)	0.45 (0.17)		
ϕ_{Core}	0.39 (0.03)	0.4 (0.03)		
ϕ_{SVN}	0.33 (0.05)	0.34 (0.06)		
ϕ^{GIV}			0.42 (0.03)	0.31 (0.06)
Tests (p -values):				
Specification	0.93	0.952		
Homogeneity	<.001	<.001		
First-stage F -statistic			411	112

Table 6: Estimation results. Coefficient estimates for $\phi_S, \phi_{IRL,PRT,ESP}, \phi_{GRC,ITA}, \phi_{Core}$ and ϕ_{SVN} are listed above standard errors, which are provided in parentheses. p values are provided in the bottom section of the table for the specification test and parameter homogeneity tests.

observed explanatory variables.

The below identification lemma (Lemma 5) extends NM Lemma 2.3, the consistency theorem (Theorem 5) extends NM Lemma 2.6, and the asymptotic normality theorem (Theorem 6) extends NM Theorem 3.4. The proofs for these extensions are, for the most part, identical to the original NM results. Notably, care is taken to ensure uniform convergence of the GMM objective function in Theorem 5 and the expected Jacobian matrices in Theorem 4.

These results can be applied to estimators where the second step GMM estimator is over-identified, as is the case for RGIV when $n \geq 4$. In contrast, an application of the results featured in Newey and McFadden (1994) Chapter 6 requires the number of moment conditions to equal the number of parameters of interest.

Lemma 5 (Identification, two-step). *If W is positive semi-definite and, for $g_0(\boldsymbol{\theta}) = \mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)]$, $g_0(\boldsymbol{\theta}_0) = 0$, and $Wg_0(\boldsymbol{\theta}) \neq 0$ for $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$ then $Q_0(\boldsymbol{\theta}) = -g_0(\boldsymbol{\theta})'Wg_0(\boldsymbol{\theta})$ has a unique maximum at $\boldsymbol{\theta}_0$.*

Proof. Apply an identical argument as the one used for the proof of NM Lemma 2.3.

Let R be such that $R'R = W$. If $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, then $0 \neq Wg_0(\boldsymbol{\theta}) = R'Rg_0(\boldsymbol{\theta})$ implies $Rg_0(\boldsymbol{\theta}) \neq 0$ and hence $Q_0(\boldsymbol{\theta}) = -[Rg_0(\boldsymbol{\theta})]'[Rg_0(\boldsymbol{\theta})] < Q_0(\boldsymbol{\theta}_0) = 0$ for $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$. \square

Theorem 5 (Consistency, two-step). *Suppose that $\mathbf{z}_i, (i = 1, 2, \dots)$, are i.i.d., $\widehat{W} \xrightarrow{p} W$, $\widehat{\boldsymbol{\gamma}} \xrightarrow{p} \boldsymbol{\gamma}_0$, and (i) W is positive semidefinite and $W\mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)] = 0$ only if $\boldsymbol{\theta} = \boldsymbol{\theta}_0$; (ii) $\boldsymbol{\theta}_0 \times \boldsymbol{\gamma}_0 \in \boldsymbol{\Theta} \times \boldsymbol{\Gamma}$ compact; (iii) $g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma})$ is continuous at each $\boldsymbol{\theta} \times \boldsymbol{\gamma} \in \boldsymbol{\Theta} \times \boldsymbol{\Gamma}$ with probability one; (iv) $\mathbb{E}[\sup_{\boldsymbol{\theta}, \boldsymbol{\gamma}} \|g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma})\|] < \infty$. Then $\widehat{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}_0$.*

Proof. Proceed by verifying the hypotheses of Theorem 2.1 of Newey McFadden (abbreviated as NM2.1):

- NM2.1(i): Follows from Lemma 5 and Condition (i).
- NM2.1(ii): Follows from Condition (ii).
- NM2.1(iii): Newey and McFadden Lemma 2.4, Condition (iii), and Condition (iv) imply that $\sup_{\boldsymbol{\theta}} \|\frac{1}{n} \sum_{i=1}^n g(\mathbf{z}_i, \boldsymbol{\theta}; \widehat{\boldsymbol{\gamma}}) - \mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \widehat{\boldsymbol{\gamma}})]\| \xrightarrow{p} 0$ and $\mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)]$ is continuous in $\boldsymbol{\theta}$. Moreover, $\widehat{\boldsymbol{\gamma}} \xrightarrow{p} \boldsymbol{\gamma}_0$ implies $\sup_{\boldsymbol{\theta}} \|\frac{1}{n} \sum_{i=1}^n g(\mathbf{z}_i, \boldsymbol{\theta}; \widehat{\boldsymbol{\gamma}}) - \mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)]\| \xrightarrow{p} 0$. Hence, 2.1(iii) holds because $Q_0(\boldsymbol{\theta}) = -\mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)]'W\mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)]$ is continuous.
- NM2.1(iv): For uniform convergence of the objective function $\widehat{Q}_n(\boldsymbol{\theta})$ to $Q_0(\boldsymbol{\theta})$, follow

similar computations to Newey and McFadden (1994) Theorem 2.6:

$$\begin{aligned}
|\widehat{Q}_n(\boldsymbol{\theta}) - Q_0(\boldsymbol{\theta})| &\leq \left\| \frac{1}{n} \sum_{i=1}^n g(\mathbf{z}_i, \boldsymbol{\theta}; \widehat{\boldsymbol{\gamma}}) - \mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)] \right\|^2 \|\widehat{W}\| + \\
&\quad 2 \left\| \frac{1}{n} \sum_{i=1}^n g(\mathbf{z}_i, \boldsymbol{\theta}; \widehat{\boldsymbol{\gamma}}) - \mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)] \right\| \left\| \frac{1}{n} \sum_{n=1}^N g(\mathbf{z}_t, \boldsymbol{\theta}; \widehat{\boldsymbol{\gamma}}) - \mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)] \right\| \|\widehat{W}\| + \\
&\quad \|\mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma}_0)]\|^2 \|\widehat{W} - W\|^2.
\end{aligned}$$

□

Theorem 6 (NM (1994) Theorem 3.4, modified). *Suppose the hypotheses of Theorem 5 are satisfied. Also assume $\sqrt{n}(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) = O_p(1)$, $\mathbb{E}[\nabla_{\boldsymbol{\gamma}} g(\mathbf{z}, \boldsymbol{\theta}_0; \boldsymbol{\gamma}_0)] = \mathbf{0}$, (i) $\boldsymbol{\theta}_0 \times \boldsymbol{\gamma}_0 \in \text{interior}(\boldsymbol{\Theta} \times \boldsymbol{\Gamma})$; (ii) $g(\mathbf{z}, \boldsymbol{\theta}; \boldsymbol{\gamma})$ is continuously differentiable in a neighborhood \mathcal{N} of $\boldsymbol{\theta}_0 \times \boldsymbol{\gamma}_0$ with probability approaching 1; (iii) $\mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}_0; \boldsymbol{\gamma}_0)] = 0$ and $\mathbb{E}[\|g(\mathbf{z}, \boldsymbol{\theta}_0; \boldsymbol{\gamma}_0)\|^2] < \infty$, (iv) $\mathbb{E}[\sup_{\boldsymbol{\theta} \times \boldsymbol{\gamma}} \|\nabla_{\boldsymbol{\theta} \times \boldsymbol{\gamma}} g(\mathbf{z}, \boldsymbol{\theta}, \boldsymbol{\gamma})\|] < \infty$; (v) $G'WG$ is non-singular for $G = \mathbb{E}[\nabla_{\boldsymbol{\theta}} g(\mathbf{z}, \boldsymbol{\theta}_0; \boldsymbol{\gamma}_0)]$. Then, for $\Sigma = \mathbb{E}[g(\mathbf{z}, \boldsymbol{\theta}_0; \boldsymbol{\gamma}_0)g(\mathbf{z}, \boldsymbol{\theta}_0; \boldsymbol{\gamma}_0)']$,*

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}(0, (G'WG)^{-1}G'W\Sigma WG(G'WG)^{-1}).$$

Proof. Conditions (i),(ii), and (iii) imply that the first order condition is satisfied with probability approaching one: $2\widehat{G}_n(\widehat{\boldsymbol{\theta}})'\widehat{W}\widehat{g}_n(\widehat{\boldsymbol{\theta}}) = 0$ for $\widehat{G}_n(\widehat{\boldsymbol{\theta}}) = \nabla_{\boldsymbol{\theta}} \frac{1}{n} \sum_{i=1}^n g(\mathbf{z}_i, \widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\gamma}})$. Expanding the FOC about $\boldsymbol{\theta}_0$ and $\boldsymbol{\gamma}_0$ and rearranging gives

$$\begin{aligned}
\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &= -(\widehat{G}_n(\widehat{\boldsymbol{\theta}})'\widehat{W}\widehat{G}_n(\overline{\boldsymbol{\theta}}))^{-1}\widehat{G}_n(\widehat{\boldsymbol{\theta}})\widehat{W}\frac{\sqrt{n}}{n}\sum_{i=1}^n g(\mathbf{z}_i, \boldsymbol{\theta}_0; \widehat{\boldsymbol{\gamma}}) \\
&= -(\widehat{G}_n(\widehat{\boldsymbol{\theta}})'\widehat{W}\widehat{G}_n(\overline{\boldsymbol{\theta}}))^{-1}\widehat{G}_n(\widehat{\boldsymbol{\theta}})\widehat{W}\frac{\sqrt{n}}{n}\sum_{i=1}^n g(\mathbf{z}_i, \boldsymbol{\theta}_0; \boldsymbol{\gamma}_0) \\
&\quad - (\widehat{G}_n(\widehat{\boldsymbol{\theta}})'\widehat{W}\widehat{G}_n(\overline{\boldsymbol{\theta}}))^{-1}\widehat{G}_n(\widehat{\boldsymbol{\theta}})\widehat{W}\frac{1}{n}\sum_{i=1}^n \nabla_{\boldsymbol{\gamma}} g(\mathbf{z}_i, \boldsymbol{\theta}_0; \overline{\boldsymbol{\gamma}})\sqrt{n}(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) \\
&\xrightarrow{p} \mathcal{N}(0, (G'WG)^{-1}G'W\Sigma WG(G'WG)^{-1})
\end{aligned}$$

where $\overline{\boldsymbol{\theta}}$ and $\overline{\boldsymbol{\gamma}}$ are intermediate values. In the above display, $\widehat{\boldsymbol{\gamma}} \xrightarrow{p} \boldsymbol{\gamma}_0$, $\widehat{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}_0$, and condition (iv) imply $\widehat{G}_n(\widehat{\boldsymbol{\theta}}) \xrightarrow{p} G$, $\widehat{G}_n(\overline{\boldsymbol{\theta}}) \xrightarrow{p} G$, and $\frac{1}{n} \sum_{i=1}^n \nabla_{\boldsymbol{\gamma}} g(\mathbf{z}_i, \boldsymbol{\theta}_0; \overline{\boldsymbol{\gamma}}) \xrightarrow{p} \mathbf{0}$. The final line follows from the Slutsky theorem and an i.i.d. central limit theorem.

□

B.4 RGIV when the shock factor is determined by known observables

In this section, I show that differencing RGIV moments can accommodate the case where the shock factor structure is entirely determined by known observables. I also show that the identification reduces to an overdetermined system of nonlinear equation, which admits a unique solution (with the parameter space restriction $\phi_S < 1$) outside of knife-edge cases.

B.4.1 Setup

Take Assumption 1 and augment with latent factors $\boldsymbol{\eta}_t$ ($m \times 1$) where $\mathbb{E}[\boldsymbol{\eta}_t \boldsymbol{\eta}_t'] = \mathbf{I}$ (and finite fourth moments) and loadings $\boldsymbol{\lambda}_i$ ($m \times 1$), which are entirely determined by observables \mathbf{x}_i ($m \times 1$) through $\boldsymbol{\Pi}$ ($m \times m$)

$$r_{it} = \phi_i r_{St} + \underbrace{\boldsymbol{\lambda}_i' \boldsymbol{\eta}_t + u_{it}}_{v_{it}}, \quad \boldsymbol{\lambda}_i = \boldsymbol{\Pi} \mathbf{x}_i.$$

Also suppose there are more GMM moment conditions (less the square of the number of observables) than there are unknowns ($n(n-1)/2 - m^2 \geq n$). $\boldsymbol{\Pi}$ ($m \times m$) maps the unit-specific observables to the loadings and is unknown to the researcher. In matrix form, for $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n]$ and $\boldsymbol{\Lambda} = [\boldsymbol{\lambda}_1 \ \boldsymbol{\lambda}_2 \ \dots \ \boldsymbol{\lambda}_n]$,

$$\boldsymbol{\Lambda} = \boldsymbol{\Pi} \mathbf{X}, \quad \mathbf{r}_t = \phi r_{St} + \underbrace{\boldsymbol{\Lambda}' \boldsymbol{\eta}_t + \mathbf{u}_t}_{\mathbf{v}_t}, \quad \text{and } \text{rank}(\mathbf{X}) = m.$$

The shock covariance matrix is $\mathbb{E}[\mathbf{v}_t \mathbf{v}_t'] = \mathbf{X}' \boldsymbol{\Pi}' \boldsymbol{\Pi} \mathbf{X} + \mathbf{D}$ for diagonal matrix \mathbf{D} with diagonal entries $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$. Vectorizing and selecting the off-diagonal elements of the covariance matrix with selection matrix \mathbf{S} ($n(n-1)/2 \times n^2$),

$$\mathbf{S} \text{vec}(\mathbb{E}[\mathbf{v}_t \mathbf{v}_t']) = \mathbf{S} (\mathbf{X}' \otimes \mathbf{X}') \text{vec}(\boldsymbol{\Pi}' \boldsymbol{\Pi}).$$

Premultiplying the above display by the annihilator matrix (for arbitrary matrix \mathbf{W} , defined as $\mathbf{W} M_{\mathbf{W}} \equiv \mathbf{I} - \mathbf{W}(\mathbf{W}' \mathbf{W})^{-1} \mathbf{W}'$),

$$\mathbf{M}_{\mathbf{S}(\mathbf{X}' \otimes \mathbf{X}')} \mathbf{S} \text{vec}(\mathbb{E}[\mathbf{v}_t \mathbf{v}_t']) = \mathbf{0}. \tag{13}$$

Hence, a linear combination of the off-diagonal elements of $\mathbb{E}[\mathbf{v}_t \mathbf{v}_t']$ equals zero.

B.4.2 Identification

Equation 13 sets a linear combination of RGIV moments to zero. Below, I show that the spillover coefficients are generally identified outside of a set of knife-edge DGPs. Equation 13 implies a system of (univariate) quadratic equations in the errors of a particular spillover coefficient. For there to exist a solution other than the true solution, these quadratic equations must have identical zeros—which isn't true in general.

For true spillover coefficients ϕ_i and proposed solutions $\tilde{\phi}_i$, let $\check{\phi}_i \equiv \phi_i - \tilde{\phi}_i$ give the error in spillover coefficient ϕ_i . The moment condition corresponding to the idiosyncratic shocks of units i and j can be written as

$$\mathbb{E} \left[(r_{it} - \tilde{\phi}_i r_{St})(r_{jt} - \tilde{\phi}_j r_{St}) \right] = \check{\phi}_i \check{\phi}_j k + \check{\phi}_i k_j + \check{\phi}_j k_i$$

for $k \equiv \mathbb{E}[r_{St}^2]$ and $k_i \equiv \mathbb{E}[r_{St}(u_{it} + \boldsymbol{\lambda}'_i \boldsymbol{\eta}_t)]$. For $i \neq j$, the above display's moments can be stored in an $n(n-1)/2 \times 1$ vector $\tilde{\mathbf{g}}$ matching the ordering of $\text{Svec}(\mathbb{E}[\mathbf{v}_t \mathbf{v}'_t])$ in Equation 13. Hence, identification of the moments characterized by Equation 13 reduces to studying the roots of $\mathbf{M}_{\mathbf{S}(\mathbf{X}' \otimes \mathbf{X}')} \tilde{\mathbf{g}} = 0$.

Since $\text{rank}(\mathbf{M}_{\mathbf{S}(\mathbf{X}' \otimes \mathbf{X}')}) = n(n-1)/2 - m^2$, $\mathbf{M}_{\mathbf{S}(\mathbf{X}' \otimes \mathbf{X}')}$ can be expressed in reduced row echelon form (after elementary row operations)

$$\begin{bmatrix} \mathbf{I}_{n(n-1)/2-m^2} & -\mathbf{A} \\ \mathbf{0}_{m^2 \times (n(n-1)/2-m^2)} & \mathbf{0}_{m^2 \times m^2} \end{bmatrix}.$$

Letting $\tilde{\mathbf{g}}^I$ be the first $n(n-1)/2 - m^2$ moments and $\tilde{\mathbf{g}}^{II}$ be the last m^2 moments, $\mathbf{M}_{\mathbf{S}(\mathbf{X}' \otimes \mathbf{X}')} \tilde{\mathbf{g}} = 0$ can equivalently be expressed as

$$\tilde{\mathbf{g}}^I = \mathbf{A} \tilde{\mathbf{g}}^{II}. \quad (14)$$

The moments $\tilde{\mathbf{g}}^I$ equal some (known) linear combination of $\tilde{\mathbf{g}}^{II}$ determined by matrix \mathbf{A} .

To characterize the solutions, fix $\tilde{\mathbf{g}}^{II}$ and let $\mathbf{a} \equiv \mathbf{A} \tilde{\mathbf{g}}^{II}$ where a_{ij} is the entry of vector \mathbf{a} that corresponds to the moment condition for shocks $i \neq j$. Then the row of Equation 14 corresponding to the shocks of units i and j is

$$\check{\phi}_i(k_j + \check{\phi}_j k) + \check{\phi}_j k_i = a_{ij}.$$

After rearranging and setting unit 1 as the reference unit, $\check{\phi}_i = \frac{a_{i1} - \check{\phi}_1 k_i}{k_1 + \check{\phi}_1 k}$ and $\check{\phi}_j = \frac{a_{j1} - \check{\phi}_1 k_j}{k_1 + \check{\phi}_1 k}$.

Substituting $\check{\phi}_i$ and $\check{\phi}_j$,

$$a_{ij} = \check{\phi}_i(k_j + \check{\phi}_j k) + \check{\phi}_j k_i = \frac{a_{i1} - \check{\phi}_1 k_i}{k_1 + \check{\phi}_1 k} k_j + \frac{a_{j1} - \check{\phi}_1 k_j}{k_1 + \check{\phi}_1 k} k_i + \frac{(a_{i1} - \check{\phi}_1 k_i)(a_{j1} - \check{\phi}_1 k_j)}{(k_1 + \check{\phi}_1 k)^2} k.$$

Rearranging yields a quadratic equation in $\check{\phi}_1$

$$0 = \alpha_1^{(ij)} \check{\phi}_1^2 + \alpha_2^{(ij)} \check{\phi}_1 + \alpha_3^{(ij)} \quad (15)$$

where $\alpha_1^{(ij)} \equiv a_{ij} k^2 + k_i k_j k$, $\alpha_2^{(ij)} \equiv 2a_{ij} k_1 k + 2k_i k_j k_1$, and $\alpha_3^{(ij)} \equiv a_{ij} k_1^2 - (a_{i1} k_j + a_{j1} k_i) k_1 - a_{i1} a_{j1} k$. Hence, the above display gives a system of quadratic equations (in $\check{\phi}_1$) defined by the pairs $i \neq j$ contained in moments $\tilde{\mathbf{g}}^I$.

If $\tilde{\mathbf{g}}^{II} = \mathbf{0}$, then $\mathbf{a} = \mathbf{0}$, $\alpha_1^{(ij)} = k_i k_j k$, $\alpha_2^{(ij)} = 2k_i k_j k_1$, and $\alpha_3^{(ij)} = 0$. Here, $0 = \check{\phi}_1(k\check{\phi}_1 + 2k_1)$. There are two solutions, which both do not depend on the index i, j : (1) the true solution $\check{\phi}_1 = 0$ and (2) a false solution $\check{\phi}_1 = -2\frac{k_1}{k}$ (which can be ruled out from the condition $\phi_S < 1$ as is done in Lemma 1).

Now instead suppose $\tilde{\mathbf{g}}^{II} \neq \mathbf{0}$. From the quadratic formula, the roots of Equation 15 (dividing by $\alpha_1^{(ij)}$) are determined by $\frac{\alpha_2^{(ij)}}{\alpha_1^{(ij)}}$ and $\frac{\alpha_3^{(ij)}}{\alpha_1^{(ij)}}$. $\frac{\alpha_3^{(ij)}}{\alpha_1^{(ij)}}$ in particular depends on the units i, j through terms k_i and k_j . Hence, besides knife-edge cases, it is not generally true that the roots for Equation 15 coincide when $\tilde{\mathbf{g}}^{II} \neq \mathbf{0}$.

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